Localized Stability Certificates for Spatially Distributed Systems over Sparse Proximity Graphs

Nader Motee† and Qiyu Sun‡

Abstract—We propose localized conditions to check exponential stability of spatially distributed linear systems. This paper focuses on systems whose coupling structures are defined using a geodesic on proximity communication graphs. We reformulate the exponential stability condition in the form of a feasibility condition that is amenable to localized implementations. Using finite truncation techniques, we obtain decentralized necessary and sufficient stability certificates. In order to guarantee global stability, it suffices to certify localized conditions over a graph covering, where the computational complexity of verifying each localized certificate is independent of network size. Then, we analyze linear networks with symmetric state-space matrices. Several robustness conditions against local matrix perturbations are obtained that are useful for tuning network parameters in a decentralized manner while ensuring global stability.

I. INTRODUCTION

The interest in stability verification of distributed and networked control systems dates back to a few decades ago. In the context of infinite-dimensional systems, the existing results in the literature is limited to characterization of stability conditions in the form of global (centralized) certificates [1]–[4]. The ongoing research in the context of finite-dimensional systems is mainly focused on developing decentralized stability conditions for some particular classes of dynamical networks [5]–[12]. The stability conditions for the class of spatially invariant systems are studied in [13], where it is shown that stability conditions in space can be equivalently verified in a proper Fourier domain using standard tools. In [14], the authors use linear matrix inequalities to develop a framework to verify stability of a class of spatially invariant systems in a localized fashion. A more general methodology to study stability properties of spatially interconnected systems is proposed in [15], that does not require spatial invariance in the underlying dynamics of the system. In [16], a spatial truncation technique is offered to check stability of a class of spatially decaying systems using covering Lyapunov equations. In [17], the authors consider robust stability analysis of sparsely interconnected networks by modeling couplings among the subsystems with integral quadratic constraints. They show that robust stability analysis of these networks can be performed by solving a set of sparse linear matrix inequalities. The string stability of a platoon of vehicles is studied in [18], where the

authors extend the well-known string stability conditions for linear cascaded networks to nonlinear settings. In [12], the problem of designing decentralized control laws using local subsystem models is addressed, where their approach allows decentralized control design in subsystem level using standard robust control techniques. As it is discussed in [12], analysis based on their results may result in quite conservative stability conditions. The authors of [19] propose an approach based on quadratic invariance, where one needs to verify stability conditions in a centralized manner. In [20], a localized and scalable algorithm to solve a class of constrained optimal control problems for discrete-time linear systems is proposed that uses a system level synthesis framework. The authors define some notions of separability that allow parallel implementation of their algorithm. Almost all these previous works deal with synthesizing a linear network using decentralized sufficient conditions. In this paper, we analyze autonomous networks, where it is assumed that a feedback control law has been already applied and network is operating in closed-loop. We propose necessary and sufficient conditions for localized verification of exponential stability of a class of finite- and infinite-dimensional spatially distributed systems. Preliminary versions of some of our results, without proofs and simulations, were presented in [21], [22]. This submission contains several new results and extensions with respect to its conference versions.

The recent works [23]–[25] suggest that stability of the class of spatially decaying systems can be verified in a localized manner using spatial truncation techniques. In this work, our focus is on a class of spatially distributed systems whose subsystems are distributed over a spatial domain and communicate with each other via signal broadcasting within a finite range. Two subsystems can communicate with each other if their spatial distance is less than their communication range. Spatially distributed proximity graphs are employed to model the communication graph of this class of dynamical networks; see Figure 4 for an illustrate example. The state matrix of a network in this class is a band (sparse) matrix, which depends on the underlying communication graph of the network. In Section IV, we formally define the class of proximity graphs and introduce a notion of spatial coverings for their corresponding networks. For a given covering of a network, we identify a (small) subset of all subsystems, so called leading subsystems, that plays a crucial role in the verification of localized stability conditions. In Section V, we provide several centralized quantitative characterizations of exponential stability property of spatially distributed linear systems. It is shown that some of these characterizations

† The author is with the Department of Mechanical Engineering and Mechanics, Packard Laboratory, Lehigh University, Bethlehem, PA. Email address: `motee@lehigh.edu`.
‡ The author is with the Department of Mathematics, University of Central Florida, Orlando FL. Email address: `qiyu.sun@ucf.edu`.

are more amenable to localized and decentralized verification schemes. Several necessary conditions for exponential stability are obtained in Section VI that can be verified in a localized manner. We provide a formula for each subsystem to compute a local stability threshold using local information, which are estimates for the global stability threshold. In general, our proposed necessary conditions are conservative, which is a reasonable price for localizing stability conditions. In Section VII, it is shown that global stability of a network can be guaranteed by only verifying a set of localized sufficient conditions in vicinity of leading subsystems. We prove that these sufficient conditions are also necessary and almost optimal. The significant feature of our localized verifiable conditions is that they depend only on the spatially localized portions of the state matrix of the system and they are independent of the size of the entire system. The sufficient conditions in Section VII are pivotal for the design of robust spatially distributed network against new adjustment (e.g., eliminating an existing or adding a new communication link) in subnetwork level as it suffices to verify stability conditions for those affected parts of network.

The sufficient conditions in Theorem 8.4 provide a reliable tool to re-examine exponential stability of a symmetric linear system on a spatially distributed network when communication/transmission links between some subsystems are lost or added, as it suffices to verify localized stability conditions for affected subsystems. In Section VIII, we show that our proposed necessary and sufficient conditions take a more tractable form for symmetric linear networks. It is proven that the global stability threshold of a symmetric linear network can be enhanced by improving the localized stability threshold via adjusting components of properly localized portions of the state matrix. In Section IX, we show how one can design a new symmetric linear network via adjusting coupling weights (i.e., elements of the state matrix) in a localized fashion. We support our theoretical findings by considering a thorough example in Section X.

The outstanding feature of our proposed framework is that it provides necessary and sufficient conditions for stability verification of both finite- and infinite-dimensional linear networks that defined over discrete spatial networks. The computational complexity of our proposed sufficient conditions does not depend on network size and one only needs to verify them for leading subsystems. In Section XI, we discuss that our proposed methodology can be applied to other related research fields, including stability analysis of systems with linear partial differential equation (PDE) models and spatially distributed nonlinear models.

II. MATHEMATICAL NOTATIONS

The set of nonnegative integers, nonnegative real numbers and complex numbers with nonnegative real parts are shown by $\mathbb{Z}_+$, $\mathbb{R}_+$ and $\mathbb{C}_+$, respectively. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are represented by $\Re(z)$ and $\Im(z)$ respectively. For a matrix $\mathbf{A}$ with complex entries, we denote its Hermitian by $\mathbf{A}^*$ and we define its Hermitian and skew-

Hermitian matrix decomposition by

$$\mathbf{A} = \mathbf{A}_h + \mathbf{A}_{ah},$$

where $\mathbf{A}_h = (\mathbf{A} + \mathbf{A}^*)/2$ and $\mathbf{A}_{ah} = (\mathbf{A} - \mathbf{A}^*)/2$. For a countable set $\mathcal{V}$, let $\ell^p(\mathcal{V}), 1 \le p \le \infty$, contain all vectors $\mathbf{c} = [c(i)]_{i \in \mathcal{V}}$ with bounded norm

$$\|\mathbf{c}\|_p := \begin{cases} \left(\sum_{i \in \mathcal{V}} |c(i)|^p \right)^{1/p} & \text{if } 1 \le p < \infty, \\ \sup_{i \in \mathcal{V}} |c(i)| & \text{if } p = \infty. \end{cases}$$

Whenever it is not ambiguous, we simply use abbreviated notation ℓ^p to represent the linear space $\ell^p(\mathcal{V})$. The set $\mathcal{B}^p, 1 \le p \le \infty$, contains all matrices $\mathbf{A}$ on ℓ^p with bounded induced norm

$$\|\mathbf{A}\|_{\mathcal{B}^p} := \sup_{\|\mathbf{c}\|_p = 1} \|\mathbf{A}\mathbf{c}\|_p.$$

For a set F, we denote its cardinality by $\#F$ and define its characteristic function by

$$\chi_F(s) := \begin{cases} 1 & \text{if } s \in F, \\ 0 & \text{otherwise.} \end{cases}$$

III. PROBLEM STATEMENT

The communication topology of a spatially distributed network can be described by a spatial (possibly infinite) graph

$$\mathcal{G} := (\mathcal{V}, \mathcal{E}), \tag{1}$$

where $\mathcal{V}$ is the set of nodes (also known as vertices) and $\mathcal{E}$ is the set of edges. Every node in the graph $\mathcal{G}$ corresponds to a subsystem and every edge represents a direct communication link between those subsystems at the two ends of that edge. In this paper, we consider networks whose subsystems communicate with each other through broadcasting within finite signal range. Therefore, two subsystems can communicate with each other if their distance in the spatial domain is not greater than their communication range. It is assumed that all subsystems are using identical communication modules and that all of them have identical communication range. The resulting communication graph for this class of networks can be modeled by spatially distributed proximity graphs.

Assumption 3.1: Throughout this paper, all communication graphs are undirected and unweighted.

The above assumption implies that communication links are bidirectional and all subsystems have identical communication capabilities.

Definition 3.2: For an undirected and unweighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, define a geodesic distance $\rho : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}_+ \cup \{+\infty\}$ by imposing that:
 (i) $\rho(i, i) = 0$ for all $i \in \mathcal{V}$;
 (ii) $\rho(i, j)$ is the number of edges in a shortest path connecting two distinct nodes $i, j \in \mathcal{V}$; and
 (iii) $\rho(i, j) = +\infty$ if there is no paths connecting distinct nodes $i, j \in \mathcal{V}$.

A geodesic distance on a graph ([26]) can be utilized to assess communication cost between two given subsystems. When two subsystems are not neighbors (i.e., not connected
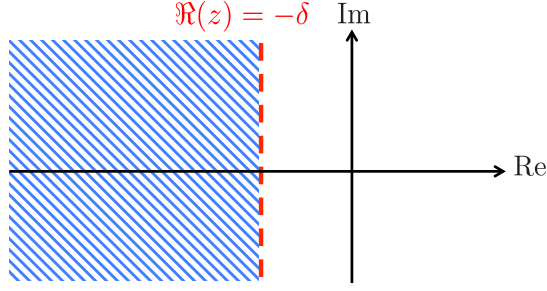
Fig. 1: Illustration of the spectral condition (7) for an exponentially stable linear system (5) whose spectrum lies somewhere in the blue shaded region for some $\delta > 0$.

Fig. 2: Illustration of 2-neighborhood of subsystem i, marked by red color, on a generic communication graph.

through a direct link), they can still communicate with each other via a chain of intermediate subsystems that connect them over a shortest path, however the communication cost between two subsystems becomes higher when their geodesic distance is bigger.

Definition 3.3: For a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a nonnegative integer τ, a matrix $\mathbf{A} = [a(i,j)]_{i,j \in \mathcal{V}}$ is called τ-banded if its entries satisfy

$$a(i,j) = 0 \quad \text{if} \quad \rho(i,j) > \tau. \qquad (2)$$

The set of all τ-banded matrices is represented by $\mathcal{B}_\tau(\mathcal{G})$ or the abbreviated notation $\mathcal{B}_\tau$ whenever it is not ambiguous.

For a matrix $\mathbf{A} = [a(i,j)]_{i,j \in \mathcal{V}}$, we define its boundedness norm by

$$\|\mathbf{A}\|_\infty := \sup_{i,j \in \mathcal{V}} |a(i,j)|$$

and its Schur norm by

$$\|\mathbf{A}\|_\mathcal{S} := \max \left\{ \sup_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} |a(i,j)|, \ \sup_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} |a(i,j)| \right\}. \qquad (3)$$

It has been proved that the following hold,

$$\|\mathbf{A}\|_\infty \leq \|\mathbf{A}\|_{\mathcal{B}^p} \leq \|\mathbf{A}\|_\mathcal{S} \leq D_1(\mathcal{G}) (1+\tau)^d \|\mathbf{A}\|_\infty \qquad (4)$$

for all matrices $\mathbf{A} \in \mathcal{B}_\tau \cap \mathcal{B}^p$, where $1 \leq p \leq \infty$ [27]. Therefore, a matrix with finite bandwidth is bounded on $\ell^p, 1 \leq p \leq \infty$, if and only if it has bounded entries.

The focus of this paper is on the following class of finite- or infinite-dimensional linear dynamical systems

$$\frac{d}{dt} \psi(t) = \mathbf{A}\psi(t) \qquad (5)$$

with initial condition $\psi(0) \in \ell^p$ for some $1 \leq p \leq \infty$, where $\psi(t) = [\psi_i(t)]_{i \in \mathcal{V}}$ and the state matrix $\mathbf{A} = [a(i,j)]_{i,j \in \mathcal{V}}$ is time independent and belongs to $\mathcal{B}_\tau$ for some finite bandwidth $\tau \geq 0$. A practical implication of having a τ-banded state matrix is that subsystems with geodesic distance greater than τ do not have a direct impact on each other's dynamics.
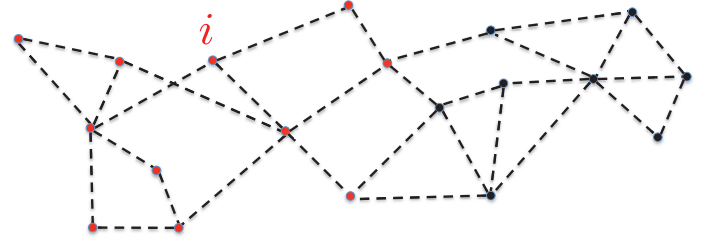
The exponential stability of the linear system (5), i.e., there exist strictly positive constants E and α such that

$$\|\psi(t)\|_2 \ \leq \ E\, e^{-\alpha t} \, \|\psi(0)\|_2 \quad \text{for all} \quad t \geq 0, \qquad (6)$$

is one of fundamental and widely studied subjects in distributed control systems literatures, see [3], [13], [28]–[30] and references therein. Recall that the solution of the linear system (5) is given by

$$\psi(t) = e^{\mathbf{A}t}\psi(0), \ t \geq 0.$$

Then its exponential stability is guaranteed by requiring the spectrum of the state matrix $\mathbf{A}$ to lie strictly in the left-half complex plane, i.e.,

$$\sigma(\mathbf{A}) \subseteq \big\{ z \in \mathbb{C} \mid \Re(z) \leq -\delta \big\} \qquad (7)$$

for some $\delta > 0$; see Figure 1 for an illustration.

The objective of this paper is to characterize spatially localized conditions, in the form of necessary and sufficient conditions, to verify the exponential stability condition (7) for both finite- and infinite-dimensional systems. This is particularly relevant to the following practical problem in analysis and design of networked systems: how do localized modifications (e.g., adding new or eliminating existing links in the communication graph or adjusting coupling weights in the state matrix) affect the global exponential stability? It is computationally advantageous to devise a method that allows us to localize and inspect stability only in relevant parts of a network, instead of verifying stability conditions globally. Moreover, localized stability certificates are suitable for decentralized/distributed implementations as they are only required to utilize local data.

In the following sections, we solve this problem and show that our localized stability conditions can be implemented in a decentralized/distributed manner and their computational complexity is independent of network size (e.g., see Theorems 7.1 and 8.4).

Remark 3.4: According to Assumption 3.1, the communication graph of the linear network (5) is undirected. It should be emphasized that this assumption does not imply that the state matrix is necessarily conjugate symmetric. In fact, our results hold for all linear dynamical networks with arbitrary state matrices having small bandwidth and real/complex entries.
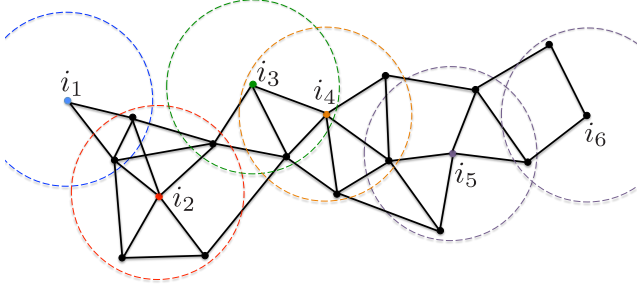
Fig. 3: A 2-covering of a spatially distributed system with 19 subsystems. This network can also be covered by 1-neighborhood of 6 leading subsystems.

IV. PROXIMITY GRAPHS AND THEIR COVERINGS

For a given communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ equipped with a geodesic distance ρ, the r-neighborhood of subsystem $i \in \mathcal{V}$ is defined by

$$B(i, r) := \{j \in \mathcal{V} \mid \rho(i, j) \leq r\},$$

see Figure 2 for an illustrative example.

In this paper, we require that the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has the following global feature: number of subsystems in the r-neighborhood and $2r$-neighborhood of each subsystem are comparable.

Assumption 4.1: The counting measure $\mu_{\mathcal{G}} : 2^{\mathcal{V}} \longmapsto \mathbb{Z}_+$ of the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a doubling measure, i.e., there exists a positive number $D_0(\mathcal{G}) \geq 1$ such that

$$\mu_{\mathcal{G}}(B(i, 2r)) \leq D_0(\mathcal{G})\, \mu_{\mathcal{G}}(B(i, r)) \tag{8}$$

hold for all $i \in \mathcal{V}$ and $r \geq 0$, where $\mu_{\mathcal{G}}(F) := \#F$ for all $F \subset \mathcal{V}$.

The minimal constant $D_0(\mathcal{G})$ for the inequality (8) to hold is known as the doubling constant of the counting measure $\mu_{\mathcal{G}}$ [27], [31]. For a communication graph $\mathcal{G}$, one can verify that the doubling constant $D_0(\mathcal{G})$ of its counting measure $\mu_{\mathcal{G}}$ dominates its maximal node degree, i.e.,

$$d_{\max}(\mathcal{G}) \leq D_0(\mathcal{G}).$$

This implies that every subsystem in a spatially distributed network, whose communication graph satisfies Assumption 4.1, communicates with at most $D_0(\mathcal{G})$ other subsystems in that network directly.

Definition 4.2: For an integer $N > 0$, an N-covering of a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a set of indices $\mathcal{V}_N = \{i_m \mid m \geq 1\}$ such that for every subsystem $i \in \mathcal{V}$ there exists at least one i_m such that $i \in B(i_m, N/2)$.

A simple procedure to identify an N-covering is by the following procedure:
  (i) taking an arbitrary subsystem $i_1 \in \mathcal{V}$ for every connected component of $\mathcal{G}$, and then
  (ii) iteratively finding new subsystems $i_m \in \mathcal{V}$ for all $m \geq 2$ such that

$$\rho(i_m, i_1) = \min \left\{ \rho(i, i_1) \,\middle|\, i \notin \bigcup_{m'=1}^{m-1} B(i_{m'}, N/2) \right\}. \tag{9}$$

The resulting N-covering $\mathcal{V}_N$ from the above algorithm satisfies the following property [27]: every subsystem $i \in \mathcal{V}$ is in the $(N/2)$-neighborhoods of $i_m \in \mathcal{V}_N$ at least once and in the $2N$-neighborhoods of $i_m \in \mathcal{V}_N$ at most $(D_0(\mathcal{G}))^5$ times, i.e.,

$$1 \leq \alpha_1 \leq \alpha_2 \leq D_0(\mathcal{G})^5 \tag{10}$$

in which

$$\alpha_1 = \sum_{i_m \in \mathcal{V}_N} \chi_{B(i_m, N/2)}(i), \tag{11}$$

$$\alpha_2 = \sum_{i_m \in \mathcal{V}_N} \chi_{B(i_m, 2N)}(i), \tag{12}$$

and $D_0(\mathcal{G})$ is the doubling constant in (8).

The set of all subsystems in $\mathcal{V}_N$ are referred to as *leading subsystems* of a spatially distributed system. The importance of leading subsystems will become more evident later in the paper, e.g., see results of Theorems 7.1, 7.2 and 8.4, where it is shown that global stability of a network can be inferred by only verifying a set of localized sufficient conditions in vicinity of leading subsystems.

The set of leading subsystems constructed through the above procedure is neither unique nor optimal, see Figure 3 for an illustrative example. Therefore, in our results, we can safely employ any subset $\tilde{\mathcal{V}}_N \subset \mathcal{V}$ that satisfies inequalities (10) as the set of leading subsystems. The number of leading subsystems in an N-covering decreases as N increases. This may impose some trade-offs between the number of leading subsystems and their minimal required on-board computational capabilities: when the number of leading subsystems decreases, they should, in turn, be equipped with more powerful computers to enable them to verify stability conditions for larger covering regions.

Definition 4.3: For a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, its counting measure $\mu_{\mathcal{G}}$ has polynomial growth if there exist positive constants $D_1(\mathcal{G})$ and d such that

$$\mu_{\mathcal{G}}(B(i, r)) \leq D_1(\mathcal{G})(1 + r)^d \tag{13}$$

for all $i \in \mathcal{V}$ and $r \geq 0$.

For a spatially distributed network with communication graph $\mathcal{G}$, the smallest constants d and $D_1(\mathcal{G})$ for which the inequality (13) holds are so called Beurling dimension and density of that network, respectively [27]. For a spatially distributed network whose subsystems are embedded on a d-dimensional manifold and direct communication link between two subsystems exists only if their spatial locations are within a certain range[1], its Beurling dimension is the same as the dimension of the manifold and its density is related to Ricci curvature of the underlying manifold.

We remark that a doubling measure $\mu_{\mathcal{G}}$ has polynomial growth,

$$\mu_{\mathcal{G}}(B(i, r)) \leq D_0(\mathcal{G})(1 + r)^{\log_2 D_0(\mathcal{G})}$$

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[1]e.g., when every subsystem communicates by broadcasting a signal using its on-board communication hardware modules.

for all $i \in \mathcal{V}$ and $r \geq 0$. However, the Beurling dimension d of the graph $\mathcal{G}$ is usually much smaller than $\log_2 D_0(\mathcal{G})$ in the above estimate.

V. CENTRALIZED EXPONENTIAL STABILITY CONDITIONS

In this section, we present several equivalent versions of the exponential stability condition (7) of a linear system on a spatially distributed proximity graph, which will be used in the next two sections to derive localized sufficient and necessary stability conditions.

Theorem 5.1: Suppose that the state matrix $\mathbf{A}$ of the linear system (5) is in $\mathcal{B}^2$. Then, its exponential stability (6) is equivalent to each of the following statements:

(i) Spectrum of the state matrix $\mathbf{A}$ is strictly contained in the open left-half complex plane, i.e.,

$$\sigma(\mathbf{A}) \subset \left\{ z \in \mathbb{C} \mid \Re(z) \leq -\delta \right\} \tag{14}$$

for some $\delta > 0$.

(ii) $z\mathbf{I} - \mathbf{A}$ is invertible for all $z \in \mathbb{C}^+$ and

$$A_0 := \inf_{z \in \mathbb{C}^+} \|(z\mathbf{I} - \mathbf{A})^{-1}\|_{\mathcal{B}^2}^{-1} > 0. \tag{15}$$

(iii) There exists a positive constant A_0 such that

$$\|(z\mathbf{I} - \mathbf{A})\mathbf{c}\|_2 \geq A_0 \|\mathbf{c}\|_2 \tag{16}$$

and

$$\|(z\mathbf{I} - \mathbf{A}^*)\mathbf{c}\|_2 \geq A_0 \|\mathbf{c}\|_2 \tag{17}$$

for all $z \in \mathbb{C}^+$ and $\mathbf{c} \in \ell^2$.

(iv) There exists a positive constant A_0 such that

$$\min \left\{ \|\mathbf{A}\mathbf{d}\|_2^2, \|\mathbf{A}^*\mathbf{d}\|_2^2 \right\} \geq A_0^2 + |\mathbf{d}^*\mathbf{A}_{ah}\mathbf{d}|^2$$
$$+ \left(\max\{0, \mathbf{d}^*\mathbf{A}_h\mathbf{d}\} \right)^2 \tag{18}$$

for all $\mathbf{d} \in \ell^2$ with $\|\mathbf{d}\|_2 = 1$.

The equivalence among the first three statements of Theorem 5.1 can be easily established, while the last statement is obtained from the third one by minimizing the left-hand expression of (16) and (17) over all $z \in \mathbb{C}^+$. For the completeness of this paper, a detailed proof of this theorem is given in Appendix A.

The constant A_0 in statements (ii), (iii), and (iv) of Theorem 5.1 can be chosen to be identical; in that case, we refer to A_0 as *stability threshold* of the linear system (5). Furthermore, it can be proven that if the statement (ii) of Theorem 5.1 holds, then the spectral set property (14) will also hold with $\delta = A_0$. However, the converse is not true in general as constant A_0 in statement (ii) may depend on $\mathbf{A}$ and its dimension (if it is finite-dimensional). For example, the square matrix $\mathbf{A} = [a(i,j)] \in \mathbb{R}^{n \times n}$ that is defined by

$$a(i,j) = \begin{cases} 1 & \text{if } j = i + 1 \\ -1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

has spectrum $\sigma(\mathbf{A}) = \{-1\}$, but the inverse matrix $\mathbf{A}^{-1} = [\check{a}(i,j)] \in \mathbb{R}^{n \times n}$ has its $\mathcal{B}^2$-norm tending to infinity as n goes

to infinity, where

$$\check{a}(i,j) = \begin{cases} -1 & \text{if } 1 \leq i \leq j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

For this reason, in order to investigate matrices with spectra contained in the open left-half complex plane, we prefer to use the bound estimate (15) instead of the spectral set property (14).

As it is shown in the following illustrative example, the stability threshold of a spatially invariant linear system can be calculated explicitly.

Example 5.2: It is well-known [3] that a spatially invariant linear system (5) with a Toeplitz state matrix

$$\mathbf{A}_0 = \left[p(i - j) \right]_{i,j \in \mathbb{Z}}$$

is exponentially stable if there exists $\delta_0 > 0$ such that

$$\Re(\hat{p}(\xi)) \leq -\delta_0$$

for all $\xi \in \mathbb{R}$, where

$$\hat{p}(\xi) = \sum_{j \in \mathbb{Z}} p(j) e^{-2\pi j \xi \sqrt{-1}}.$$

By direct computation, we have

$$\|(z\mathbf{I} - \mathbf{A}_0)\mathbf{c}\|_2 \geq \left(\inf_{\xi \in \mathbb{R}} |z - \hat{p}(\xi)| \right) \|\mathbf{c}\|_2 \tag{19}$$

and

$$\|(z\mathbf{I} - \mathbf{A}_0^*)\mathbf{c}\|_2 \geq \left(\inf_{\xi \in \mathbb{R}} |\bar{z} - \hat{p}(\xi)| \right) \|\mathbf{c}\|_2 \tag{20}$$

for all $z \in \mathbb{C}^+$ and $\mathbf{c} \in \ell^2$. From these inequalities, we can calculate the following estimate for the stability threshold A_0 of the spatially invariant linear system

$$A_0 = \inf_{z \in \mathbb{C}^+} \inf_{\xi \in \mathbb{R}} |z - \hat{p}(\xi)|, \tag{21}$$

that satisfies $A_0 \geq \delta_0$.

We finish this section with a remark on the connection between exponential stability of a linear system and matrix stability.

Remark 5.3: The notion of matrix stability for a matrix $\mathbf{B} \in \mathcal{B}^2$, i.e., there exists a positive constant E such that

$$\|\mathbf{B}\mathbf{c}\|_2 \geq E\|\mathbf{c}\|_2 \quad \text{for all} \quad \mathbf{c} \in \ell^2,$$

is one of fundamental tools in frame theory, sampling theory, wavelet analysis and many other fields [32]–[37], where matrix stability verification in a decentralized/distributed manner has been studied in [27], [38]. By Theorem 5.1, exponential stability of the linear system (5) with a state matrix $\mathbf{A}$ can be understood as uniform stability of the family of matrices $z\mathbf{I} - \mathbf{A}$ and $z\mathbf{I} - \mathbf{A}^*, z \in \mathbb{C}^+$.

VI. DECENTRALIZED NECESSARY CONDITIONS

In this section, we utilize finite truncation techniques to obtain several decentralized necessary conditions for exponential stability of the spatially distributed linear network (5). In

Section VII, it is shown that these conditions become also sufficient for large value of N.

Definition 6.1: Suppose that the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a spatially distributed network satisfies Assumption 3.1. For every node $i \in \mathcal{V}$ and integer $N \geq 0$, the truncation operator $\chi_i^N : \ell^2 \longmapsto \ell^2$ is defined by

$$\chi_i^N[c(j)]_{j\in\mathcal{V}} := [\chi_{B(i,N)}(j)\,c(j)]_{j\in\mathcal{V}}. \tag{22}$$

The truncation operator χ_i^N localizes a vector to the N-neighborhood of the subsystem $i \in \mathcal{V}$ and its action can be equivalently expressed by a diagonal matrix whose (j,j)-th diagonal entry is equal to $\chi_{B(i,N)}(j) := \chi_{[0,1]}(\rho(i,j)/N)$ for all $j \in \mathcal{V}$.

Theorem 6.2: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the communication graph of a spatially distributed network. Suppose that the state matrix $\mathbf{A}$ of the linear system (5) belongs to $\mathcal{B}_\tau \cap \mathcal{B}^2$ for some integer $\tau \geq 0$ and the system is exponentially stable with stability threshold A_0. Then, inequalities

$$\|(z\mathbf{I} - \mathbf{A})\chi_i^N \mathbf{c}\|_2 \geq A_0 \|\chi_i^N \mathbf{c}\|_2 \tag{23}$$

and

$$\|(z\mathbf{I} - \mathbf{A}^*)\chi_i^N \mathbf{c}\|_2 \geq A_0 \|\chi_i^N \mathbf{c}\|_2 \tag{24}$$

hold for all vertices $i \in \mathcal{V}$, positive integers $N \geq 1$, complex numbers $z \in \mathbb{C}^+$, and vectors $\mathbf{c} \in \ell^2$.

Proof: Conditions (23) and (24) are the localized version of (16) and (17) in Theorem 5.1. Since inequalities (16) and (17) hold for all $\mathbf{c}$ in the Hilbert space ℓ^2, they also hold for all vectors in $\{\chi_i^N \mathbf{c} \mid \mathbf{c} \in \ell^2\}$, which is a subset of ℓ^2. Moreover, the constant in (23) and (24) can be selected to be the same as the one in (16) and (17). ∎

The necessary conditions (23) and (24) are spatially localized and can be verified in a decentralized/distributed manner by having access only to local information about the state matrix $\mathbf{A}$. However, they require evaluations over all complex numbers $z \in \mathbb{C}^+$. This burden can be resolved by taking infimum of norm quantities in the left hand side of conditions (23) and (24) over all $z \in \mathbb{C}^+$.

Theorem 6.3: Let the communication graph $\mathcal{G}$, stability threshold A_0 and state matrix $\mathbf{A}$ be as in Theorem 6.2. Then, inequality

$$\min\{\|\mathbf{A}\chi_i^N \mathbf{d}\|_2^2, \|\mathbf{A}^*\chi_i^N \mathbf{d}\|_2^2\} \geq A_0^2 + \Phi_i^N(\mathbf{d}) \tag{25}$$

hold for all $N \geq 1$, $i \in \mathcal{V}$, and $\mathbf{d} \in \ell^2$ with $\chi_i^N \mathbf{d} = \mathbf{d}$ and $\|\chi_i^N \mathbf{d}\|_2 = 1$, where

$$\Phi_i^N(\mathbf{d}) = \frac{|\mathbf{d}^* \chi_i^N \mathbf{A}_{ah} \chi_i^N \mathbf{d}|^2 + \left(\max\{0, \mathbf{d}^* \chi_i^N \mathbf{A}_h \chi_i^N \mathbf{d}\}\right)^2}{\|\chi_i^N \mathbf{d}\|_2^2}.$$

Proof: It is straightforward to verify that

$$\inf_{z\in\mathbb{C}^+} \|(z\mathbf{I} - \mathbf{A})\chi_i^N \mathbf{d}\|_2^2 = \mathbf{d}^* \chi_i^N \mathbf{A}^* \mathbf{A} \chi_i^N \mathbf{d} - \Phi_i^N(\mathbf{d})$$

and

$$\inf_{z\in\mathbb{C}^+} \|(z\mathbf{I} - \mathbf{A}^*)\chi_i^N \mathbf{d}\|_2^2 = \mathbf{d}^* \chi_i^N \mathbf{A} \mathbf{A}^* \chi_i^N \mathbf{d} - \Phi_i^N(\mathbf{d}).$$

These two expressions along with the conclusions in Theorem 6.2 completes the proof. ∎

For $i \in \mathcal{V}$ and $N \geq 1$, let us define

$$B_N(i) := \inf_{\|\chi_i^N \mathbf{d}\|_2 = 1} \sqrt{\min\{\|\mathbf{A}\chi_i^N \mathbf{d}\|_2^2, \|\mathbf{A}^*\chi_i^N \mathbf{d}\|_2^2\} - \Phi_i^N(\mathbf{d})}. \tag{26}$$

Using the following relationship

$$\{\chi_i^N \mathbf{d} \mid \mathbf{d} \in \ell^2\} \subseteq \{\chi_i^{N+1} \mathbf{d} \mid \mathbf{d} \in \ell^2\} \subseteq \ell^2,$$

we have that

$$B_N(i) \geq B_{N+1}(i) \tag{27}$$

for all $N \geq 1$ and $i \in \mathcal{V}$. By Theorem 6.3, this sequence is bounded below by A_0, i.e.,

$$B_N(i) \geq A_0 \tag{28}$$

for all $N \geq 1$ and $i \in \mathcal{V}$. In fact, the sequence $B_N(i)$ decreases to A_0 as N increases for every $i \in \mathcal{V}$, i.e.,

$$\lim_{N\to\infty} B_N(i) = A_0 \quad \text{for all } i \in \mathcal{V} \tag{29}$$

by (27) and the fact that

$$\bigcup_{N\geq 1} \{\chi_i^N \mathbf{d} \mid \mathbf{d} \in \ell^2\}$$

is a dense subset of ℓ^2.

For a spatially invariant linear system with a Toeplitz state matrix $\mathbf{A}_0 = [p(i-j)]_{i,j\in\mathbb{Z}}$, the local stability threshold $B_N(i)$ in (26) is independent of the node index i. If the state matrix $\mathbf{A}_0$ is further assumed to have finite bandwidth τ, we can obtain an estimate for the local stability threshold.

Example 6.4: [Continuation of Example 5.2] For every $N \geq \tau$, $i \in \mathbb{Z}$ and $\mathbf{c} = [c(j)]_{j\in\mathbb{Z}} \in \ell^2$, we have

$$\begin{aligned}
\|(z\mathbf{I} - \mathbf{A}_0)\chi_i^N \mathbf{c}\|_2^2 &= \sum_{k=i-N-\tau}^{i+N+\tau} \left|\sum_{l\in\mathbb{Z}} p_z(k-l)c_{N,i}(l)\right|^2 \\
&= \sum_{k=i-N}^{i+N} \left|\sum_{l\in\mathbb{Z}} p_z(k-l)c_{N,i}(l)\right|^2 \\
&\quad + \sum_{k=i+N-\tau+1}^{i+N} \left|\sum_{l\in\mathbb{Z}} p_z(k-l)c_{N,i}(l-2N-1)\right|^2 \\
&\quad + \sum_{k=i-N}^{i-N+\tau-1} \left|\sum_{l\in\mathbb{Z}} p_z(k-l)c_{N,i}(l+2N+1)\right|^2,
\end{aligned}$$

where $z - \hat{p}(\xi) = \sum_{j\in\mathbb{Z}} p_z(j)e^{-2\pi\sqrt{-1}j\xi}$ and $\chi_i^N \mathbf{c} = [c_{N,i}(j)]_{j\in\mathbb{Z}}$. Therefore, we get

$$\begin{aligned}
\|(z\mathbf{I} - \mathbf{A}_0)\chi_i^N \mathbf{c}\|_2^2 &\geq \sum_{k=i-N+\tau}^{i+N-\tau} \left|\sum_{l\in\mathbb{Z}} p_z(k-l)c_{N,i}(l)\right|^2 \\
&\quad + \frac{1}{2} \sum_{k=i+N-\tau+1}^{i+N} \left|\sum_{l\in\mathbb{Z}} p_z(k-l)(c_{N,i}(l-2N-1) + c_{N,i}(l))\right|^2
\end{aligned}$$

$$+ \frac{1}{2} \sum_{k=i-N}^{i-N+\tau-1} \left| \sum_{l \in \mathbb{Z}} p_z(k-l)(c_{N,i}(l+2N+1) + c_{N,i}(l)) \right|^2$$

$$\geq \frac{1}{2} \sum_{k=i-N}^{i+N} \left| \sum_{l=i-N}^{i+N} p_z(k-l)\tilde{c}_{N,i}(l) \right|^2,$$

where $\tilde{\mathbf{c}}_{N,i} = \left[\tilde{c}_{N,i}(j) \right]_{j \in \mathbb{Z}}$ is a periodic vector taking the same values with $\chi_i^N \mathbf{c}$ on intervals $[i-N, i+N]$. Thus,

$$\left\| (z\mathbf{I} - \mathbf{A}_0)\chi_i^N \mathbf{c} \right\|_2^2 \geq \frac{\inf_{\xi \in \mathbb{Z}/(2N+1)} |z - \hat{p}(\xi)|^2}{2} \left\| \chi_i^N \mathbf{c} \right\|_2^2$$

for all $\mathbf{c} \in \ell^2$. Using the same argument, we can show that

$$\left\| (z\mathbf{I} - \mathbf{A}_0^*)\chi_i^N \mathbf{c} \right\|_2^2 \geq \frac{\inf_{\xi \in \mathbb{Z}/(2N+1)} |\bar{z} - \hat{p}(\xi)|^2}{2} \left\| \chi_i^N \mathbf{c} \right\|_2^2$$

for all $\mathbf{c} \in \ell^2$. As a result, we obtain the following estimate for the local stability threshold of the spatially invariant linear system (5),

$$B_N(i) \geq \frac{\sqrt{2}}{2} \inf_{z \in \mathbb{C}^+} \inf_{\xi \in \mathbb{Z}/(2N+1)} |z - \hat{p}(\xi)| \quad (30)$$

for all $N \geq \tau$ and $i \in \mathbb{Z}$. This together with (21) implies that

$$B_N(i) \geq \frac{\sqrt{2}}{2} A_0 \quad (31)$$

for all $N \geq \tau$ and $i \in \mathbb{Z}$, where A_0 is the stability threshold of the spatially invariant linear system. To summarize the above argument, the stability threshold A_0 is obtained in a centralized fashion using Theorem 5.1, while our estimate for the local stability threshold can be calculated via spatially localized conditions given in Theorem 6.2. By comparing (28) and (31), one observes that our decentralized and localized stability conditions provide more loose estimate for the stability threshold A_0. This is the price of going from centralized to decentralized/distributed verification.

VII. DECENTRALIZED SUFFICIENT CONDITIONS

In this section, we introduce several decentralized sufficient conditions to verify exponential stability of the linear system (5) that is defined on a spatially distributed proximity graph. Our proposed sufficient conditions are based on the limit property (29) and the fact that the localized stability thresholds are uniformly bounded below by the stability threshold, according to (28).

Theorem 7.1: Let the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a spatially distributed network satisfy Assumption 3.1 and its counting measure $\mu_{\mathcal{G}}$ have the polynomial growth (13). Suppose that the state matrix $\mathbf{A}$ belongs to $\mathcal{B}_\tau \cap \mathcal{B}^2$ for some $\tau \geq 0$. If there exists a positive integer $N_0 \geq \tau$ and a positive number B_{N_0} satisfying

$$B_{N_0} \geq 4\tau \sqrt{\frac{\alpha_2^*}{\alpha_1^*}} \|\mathbf{A}\|_{\mathcal{S}} N_0^{-1} \quad (32)$$

such that

$$\left\| (z\mathbf{I} - \mathbf{A})\chi_{i_m}^{N_0}\mathbf{c} \right\|_2 \geq B_{N_0} \left\| \chi_{i_m}^{N_0}\mathbf{c} \right\|_2 \quad (33)$$

and

$$\left\| (z\mathbf{I} - \mathbf{A}^*)\chi_{i_m}^{N_0}\mathbf{c} \right\|_2 \geq B_{N_0} \left\| \chi_{i_m}^{N_0}\mathbf{c} \right\|_2 \quad (34)$$

hold for all $z \in \mathbb{C}^+, \mathbf{c} \in \ell^2$ and $i_m \in \mathcal{V}_{N_0}$ (the set of leading subsystems according to Definition 4.2), then the linear system (5) with state matrix $\mathbf{A}$ is exponentially stable and its stability threshold A_0 satisfies

$$A_0 \geq \frac{1}{2} B_{N_0} \sqrt{\frac{\alpha_1^*}{\alpha_2^*}}, \quad (35)$$

in which

$$\alpha_1^* := \inf_{i \in \mathcal{V}} \sum_{i_m \in \mathcal{V}_{N_0}} \chi_{B(i_m, N_0/2)}(i), \quad (36)$$

$$\alpha_2^* := \sup_{i \in \mathcal{V}} \sum_{i_m \in \mathcal{V}_{N_0}} \chi_{B(i_m, 2N_0)}(i). \quad (37)$$

A detailed proof of Theorem 7.1 is included in Appendix B. The sufficient conditions (33) and (34) in Theorem 7.1, in their current form, require verification for all complex numbers $z \in \mathbb{C}^+$. In the following result, we obtain an equivalent verifiable condition by eliminating the complex variable and combining these two conditions into one.

Theorem 7.2: Suppose that all assumptions of Theorem 7.1 hold. Then the linear system (5) with state matrix $\mathbf{A} \in \mathcal{B}_\tau \cap \mathcal{B}^2$ for some $\tau \geq 1$, is exponentially stable if there exist a positive integer $N_0 \geq \tau$ and a constant B_{N_0} satisfying (32) such that

$$\min \left\{ \|\mathbf{A}\chi_{i_m}^{N_0}\mathbf{d}\|_2^2, \|\mathbf{A}^*\chi_{i_m}^{N_0}\mathbf{d}\|_2^2 \right\} \geq B_{N_0}^2 + \Phi_{i_m}^{N_0}(\mathbf{d}) \quad (38)$$

for all $i_m \in \mathcal{V}_{N_0}$ and vectors $\mathbf{d} \in \ell^2$ with $\chi_{i_m}^N \mathbf{d} = \mathbf{d}$ and $\|\chi_{i_m}^N \mathbf{d}\|_2 = 1$, where $\Phi_{i_m}^{N_0}$ is defined in Theorem 6.3.

We omit a detailed proof of this theorem as it uses similar arguments as in the proof of Theorem 6.3. The sufficient conditions in Theorems 7.1 and 7.2 assert that exponential stability can be only verified in neighborhoods of the leading subsystems, i.e., one only needs to validate the condition (38) for leading subsystems in $\mathcal{V}_{N_0}$, rather than checking them for every single subsystem. This feature drastically reduces time-complexity of the verification process and makes it attractive for real-world applications. Our results also suggest an important design protocol: all leading subsystems of a spatially distributed system should be equipped with high performance computing and communication modules to allow them to verify localized stability conditions more reliably and timely.

For the set of leading subsystems $\mathcal{V}_{N_0}$ in Theorem 7.1, the corresponding covering constants α_1, α_2 in (11) and (12) satisfy the following inequities according to (10):

$$1 \leq \alpha_1 \leq \alpha_2 \leq D_1(\mathcal{G})^5. \quad (39)$$

As a result, the right hand side of (32) tends to zero when $N_0 \to \infty$. This implies that the sufficient conditions (33) and (34) for exponential stability of the linear dynamical network

(5) is also necessary, cf. Theorem 6.2. From Theorems 7.1 and 6.2, we conclude that exponential stability of the linear system (5) can be verified via a decentralized/distributed manner. Moreover, the global stability threshold A_0 in (16) and (17) and the local stability threshold B_{N_0} in (33) and (34) are comparable through the following inequalities

$$A_0 \leq B_{N_0} \leq 2\sqrt{\frac{\alpha_2^*}{\alpha_1^*}}\, A_0 \qquad (40)$$

for those integers N_0 satisfying (32).

The implication of small N_0 in real-world applications is that the leading subsystems can be equipped with less powerful computers to verify stability conditions. The requirement (32) on size of N_0 is conservative, but it is almost optimal. In the following example, it shows that a linear network may not be exponentially stable even though conditions (33) and (34) are met with some constant B'_{N_0} that has the same order of N_0^{-1} as the lower bound in (32) for large enough N_0.

Example 7.3: Let us consider a spatially invariant system whose state matrix is a bi-infinite Toeplitz matrix $\mathbf{A}_1 = [a_1(i-j)]_{i,j\in\mathbb{Z}}$ with Fourier symbol

$$\sum_{k\in\mathbb{Z}} a_1(k)e^{-2\pi k\sqrt{-1}\xi} = -1 + e^{-2\pi\sqrt{-1}\xi}.$$

It is straightforward to check that $\mathbf{A}_1$ is 1-band matrix with $\|\mathbf{A}_1\|_\infty = 1$ and property $0 \in \sigma(\mathbf{A}_1)$. Therefore, the linear system (5) with state matrix $\mathbf{A}_1$ is not exponentially stable. In the following, we show that (33) and (34) hold for this system with constant $B'_{N_0} = \frac{1}{2}N_0^{-1}$. For every $z \in \mathbb{C}^+, i \in \mathbb{Z}$ and $N_0 \geq 1$, we have

$$\inf_{\|\chi_i^{N_0}\mathbf{c}\|_2=1} \|(z\mathbf{I} - \mathbf{A}_1)\chi_i^{N_0}\mathbf{c}\|_2^2$$
$$= \inf_{\sum_{j=1}^{2N_0+1}|d_j|^2=1} \left\{ |(z+1)d_1|^2 + |d_{2N_0+1}|^2 \right.$$
$$\left. + \sum_{j=2}^{2N_0+1} |(z+1)d_j - d_{j-1}|^2 \right\}$$
$$= \inf_{\sum_{j=1}^{2N_0+1}|d_j|^2=1} |z+1|^2 + 1$$
$$\qquad\qquad - 2\Re\left\{(z+1)\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right\}. \qquad (41)$$

Let us write $z = a + b\sqrt{-1} \in \mathbb{C}^+$. Then, we get

$$|z+1|^2 - 2\Re\left\{(z+1)\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right\}$$
$$= (a+1)^2 - 2(a+1)\Re\left\{\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right\}$$
$$+ b^2 + 2b\Im\left\{\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right\}$$

$$\geq 1 + 2\Re\left\{\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right\} - \left\{\Im\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right\}^2$$
$$= \left(1 + \Re\left\{\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right\}\right)^2 - \left|\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right|^2$$
$$\geq \left(1 - \sum_{j=2}^{2N_0+1} |d_j||d_{j-1}|\right)^2 - \left(\sum_{j=2}^{2N_0+1} |d_j||d_{j-1}|\right)^2$$
$$= 1 - 2\sum_{j=2}^{2N_0+1} |d_j||d_{j-1}|, \qquad (42)$$

where all the inequalities hold because of

$$\left|\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\right| \leq \left(\sum_{j=2}^{2N_0+1} |d_j|^2\right)^{1/2}\left(\sum_{j=2}^{2N_0+1} |d_{j-1}|^2\right)^{1/2}$$
$$\leq 1.$$

Therefore, it follows from (41) and (42) that

$$\inf_{\|\chi_i^{N_0}\mathbf{c}\|_2=1} \|(z\mathbf{I} - \mathbf{A}_1)\chi_i^{N_0}\mathbf{c}\|_2^2$$
$$\geq \inf_{\sum_{j=1}^{2N_0+1}|d_j|^2=1} 2 - 2\sum_{j=2}^{2N_0+1} |d_j||d_{j-1}|$$
$$= \inf_{\sum_{j=1}^{2N_0+1}|e_j|^2=1, e_j\in\mathbb{R}} 2 - 2\sum_{j=2}^{2N_0+1} e_je_{j-1}$$
$$= \inf_{\sum_{j=1}^{2N_0+1}|e_j|^2=1, e_j\in\mathbb{R}} |e_1|^2 + |e_{2N_0+1}|^2$$
$$\qquad\qquad + \sum_{j=2}^{2N_0+1} |e_j - e_{j-1}|^2$$
$$= 2\sin\frac{\pi}{4N_0+4} \geq \frac{1}{2}N_0^{-1}$$

for all $z \in \mathbb{C}^+, i \in \mathbb{Z}$, and $N_0 \geq 1$, where the last equality follows from [39, Lemma 1 of Chapter 9]. Following a similar argument, we have

$$\inf_{\|\chi_i^{N_0}\mathbf{c}\|_2=1} \|(z\mathbf{I} - \mathbf{A}_1^*)\chi_i^{N_0}\mathbf{c}\|_2^2$$
$$= \inf_{\sum_{j=1}^{2N_0+1}|d_j|^2=1} \left\{ |d_1|^2 + |(z+1)d_{2N_0+1}|^2 \right.$$
$$\left. + \sum_{j=1}^{2N_0} |(z+1)d_j - d_{j+1}|^2 \right\}$$
$$\geq \frac{1}{2}N_0^{-1}.$$

Thus, the conditions (33) and (34) hold with constant $B'_{N_0} = (2N_0)^{-1}$ for all $N_0 \geq 1$. On the other hand, we observe that the underlying communication graph of this system with node set $\mathcal{V} = \mathbb{Z}$ has Beurling dimension 1, density 2, and the set of leading subsystems $\mathcal{V}_{N_0} = (N_0+1)\mathbb{Z}$ with covering constants $C_1 = 1$ and $C_2 = 4$. Therefore, the lower bound for the

constant B_{N_0} in (32) is $2^6 N_0^{-1}$. One observes that although conditions (33) and (34) hold with some constant B'_{N_0}, whose value is smaller than $2^6 N_0^{-1}$ but vanishes with the same rate of N_0^{-1}, the linear system is still not exponentially stable. This explains the critical role and near-optimality of the sufficient condition (32).

VIII. SYMMETRIC LINEAR SYSTEMS

In this section, we consider exponential stability of the following linear system

$$\frac{d}{dt}\psi(t) = \mathbf{B}\psi(t) \tag{43}$$

with initial condition $\psi(0) \in \ell^2$, whose state matrix $\mathbf{B}$ is Hermitian. For example, it is straightforward to check that the linear system (5) with state matrix $\mathbf{A}$ and initial condition $\psi(0) \in \ell^2$ is exponentially stable if the linear system (43) with state matrix $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$ and initial condition $\psi(0)$ is exponentially stable. In the following, we present several equivalent conditions for exponential stability of the symmetric linear systems (43), which take more simpler forms than those conditions in Theorem 5.1.

Theorem 8.1: The exponential stability of the linear system (43) with a Hermitian state matrix $\mathbf{B}$ in $\mathcal{B}^2$ is equivalent to each of the following statements:
 (i) $\mathbf{B}$ is strictly negative definite.
 (ii) There exists a positive constant A_0 such that

$$\|(z\mathbf{I} - \mathbf{B})\mathbf{c}\|_2 \geq A_0 \|\mathbf{c}\|_2 \tag{44}$$

 for all $z \in \mathbb{C}^+$ and $\mathbf{c} \in \ell^2$.
 (iii) There exists a positive constant A_0 such that

$$\mathbf{c}^* \mathbf{B} \mathbf{c} \leq 0 \tag{45}$$

 and

$$\|\mathbf{B}\mathbf{c}\|_2 \geq A_0 \|\mathbf{c}\|_2 \tag{46}$$

 hold for all $\mathbf{c} \in \ell^2$.

Condition (46) implies that constant A_0 is equal to the absolute value of the maximal eigenvalue of negative definite matrix $\mathbf{B}$.

Example 8.2: When the linear system (5) is spatially invariant with a Hermitian Toeplitz state matrix $\mathbf{B}_0 = \left[p(i - j) \right]_{i,j\in\mathbb{Z}}$, its Fourier symbol $\hat{p}(\xi)$ becomes real-valued and takes negative values. According to (21), stability threshold of the symmetric spatially invariant linear system is equal to

$$A_0 = \inf_{\xi\in\mathbb{R}} -\hat{p}(\xi).$$

Building upon Theorem 8.1, we propose the following necessary conditions that can be verified by evaluating maximum or minimum eigenvalues of some localized matrices, cf. Theorems 6.2 and 6.3.

Theorem 8.3: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the communication graph of a spatially distributed linear network (43). Suppose that the symmetric linear system (43) is exponentially stable with stability threshold A_0, whose state matrix $\mathbf{B}$ is a Hermitian matrix in $\mathcal{B}_\tau \cap \mathcal{B}^2$ for some $\tau \geq 0$. Then, the following localized inequalities

$$\mathbf{c}^* \chi_i^N \mathbf{B} \chi_i^N \mathbf{c} \leq 0 \tag{47}$$

and

$$\mathbf{c}^* \chi_i^N \mathbf{B}^2 \chi_i^N \mathbf{c} \geq A_0^2 \|\chi_i^N \mathbf{c}\|_2^2 \tag{48}$$

hold for all $N \geq \tau$, $i \in \mathcal{V}$, and $\mathbf{c} \in \ell^2$.

Proof: It is similar to the proof of Theorem 6.3. ∎

Inequalities (47) and (48) are localized version of global stability conditions (45) and (46) in Theorem 8.1. For symmetric linear systems, sufficient conditions for the exponential stability take rather simple forms.

Theorem 8.4: Suppose that all assumptions of Theorem 7.1 hold and state matrix $\mathbf{B} \in \mathcal{B}_\tau \cap \mathcal{B}^2$ is Hermitian for some $\tau \geq 0$. Then, the linear system (43) with state matrix $\mathbf{B}$ is exponentially stable if there exists a positive integer N_0 and a positive number B_{N_0} satisfying (32) such that

$$\mathbf{c}^* \chi_{i_m}^{N_0} \mathbf{B} \chi_{i_m}^{N_0} \mathbf{c} \leq 0 \tag{49}$$

and

$$\mathbf{c}^* \chi_{i_m}^{N_0} \mathbf{B}^2 \chi_{i_m}^{N_0} \mathbf{c} \geq B_{N_0}^2 \|\chi_{i_m}^{N_0} \mathbf{c}\|_2^2 \tag{50}$$

hold for all $i_m \in \mathcal{V}_{N_0}$ and $\mathbf{c} \in \ell^2$.

Proof: For every $z \in \mathbb{C}^+$ and $\mathbf{c} \in \ell^2$, by Theorem 8.1, it suffices to prove the uniform stability for the family of matrices $z\mathbf{I} - \mathbf{B}$, i.e.,

$$\|(z\mathbf{I} - \mathbf{B})\mathbf{c}\|_2 \geq \frac{B_{N_0}}{2} \sqrt{\frac{\alpha_1^*}{\alpha_2^*}} \|\mathbf{c}\|_2. \tag{51}$$

From sufficient conditions (49) and (50), it follows that

$$\begin{aligned}
\|(z\mathbf{I} - \mathbf{B})\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 &= |z|^2 \|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \\
&\quad - 2\Re(z)\, \mathbf{c}^* \chi_{i_m}^{N_0} \mathbf{B} \chi_{i_m}^{N_0} \mathbf{c} + \|\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\|^2 \\
&\geq B_{N_0}^2 \|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2
\end{aligned}$$

for all $\mathbf{c} \in \ell^2$ and $i_m \in \mathcal{V}_{N_0}$. Applying the above estimate and using similar argument used in the proof of Theorem 7.1, one can conclude the inequality (51) for all $z \in \mathbb{C}^+$. ∎

For a given Hermitian matrix $\mathbf{B} = [b(i,j)]_{i,j\in\mathcal{V}}$ in $\mathcal{B}_\tau \cap \mathcal{B}^2$, the sufficient conditions (49) and (50) in Theorem 8.4 are spatially localized in neighborhoods of each leading subsystem $i_m \in \mathcal{V}_{N_0}$, where each leading subsystem has to only have access to localized portions of state matrix $\mathbf{B}$ determined by truncation operator $\chi_{i_m}^{N_0}$. In particular, the requirement (49) holds if the largest eigenvalue of the spatially localized principal submatrix $[b(j,j')]_{j,j'\in B(i_m,N)}$ is non-positive for every subsystem $i_m \in \mathcal{V}_{N_0}$. For a Hermitian matrix $\mathbf{B}$, let us define

$$\tilde{B}_N(i) = \inf_{\|\chi_i^N\mathbf{c}\|_2=1} \|\mathbf{B}\chi_i^N\mathbf{c}\|_2 \tag{52}$$

in which $N \geq 1$ and $i \in \mathcal{V}$. The quantity $\tilde{B}_N(i)$ is equal to the square root of the smallest eigenvalue of the spatially localized matrix

$$\chi_i^N \mathbf{B}^2 \chi_i^N = \left[\sum_{k \in B(j,\tau) \cap B(j',\tau)} b(j,k)b(k,j') \right]_{j,j' \in B(i,N)} \quad (53)$$

that can be evaluated in a decentralized/distributed manner. Then, the constant $\tilde{B}_{N_0}$ in (50) can be thought of as the uniform stability threshold for small-scale systems with state matrices $\chi_{i_m}^{N_0} \mathbf{B}^2 \chi_{i_m}^{N_0}$ for all $i_m \in \mathcal{V}_{N_0}$.

Using a similar argument that leds to (29), one can verify that $\{\tilde{B}_N(i)\}_{N=\tau}^{\infty}$ is a decreasing sequence that converges to A_0 for every $i \in \mathcal{V}$, i.e.,

$$\lim_{N \to \infty} \tilde{B}_N(i) = A_0 \quad (54)$$

for all $i \in \mathcal{V}$. Inequalities (40) and (51) imply that the global stability threshold of a symmetric linear dynamical network can be enhanced by improving the localized stability threshold via adjusting components of properly localized portions of the state matrix. The following section shows how this idea can be implemented.

IX. Design of Spatially Distributed Networks

In this section, we consider the problem of coupling weight adjustment between a given pair of subsystems in an exponentially stable symmetric linear dynamical network (43). The coupling weight between subsystems $k, l \in \mathcal{V}$ can be adjusted in a localized manner via the following class of feedback control laws

$$u(t) = w \, \mathbf{E}_{kl} \, \psi(t) \quad (55)$$

that modifies the dynamics of (43) as follows

$$\frac{d}{dt} \psi(t) = \mathbf{B} \, \psi(t) + u(t), \quad (56)$$

where w is a scalar feedback gain, $\mathbf{E}_{kl} = [e(i,j)]_{i,j \in \mathcal{V}}$, and

$$e(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \{(k,l),(l,k)\} \\ 0 & \text{otherwise.} \end{cases}$$

The conclusion of Theorem 8.4 plays a critical role in computing an admissible range of values for the scalar w such that the resulting closed-loop network

$$\frac{d}{dt} \psi(t) = (\mathbf{B} + w\mathbf{E}_{kl}) \, \psi(t) \quad (57)$$

remains exponentially stable with stability threshold equal or greater than the original network. From network design perspective, when an existing coupling between subsystems k and l satisfies $b(k,l) > 0$ in the state matrix $\mathbf{B} = [b(i,j)]_{i,j \in \mathcal{V}}$, local weight adjustment law (55) will strengthen the existing coupling when $w > 0$ and weaken the existing coupling (and possibly zero it out) whenever $w < 0$.

To state the following main result of this section, we let $\mathcal{B}_\tau(M)$ denote the set of all band matrices $\mathbf{B} \in \mathcal{B}_\tau$ with bounded entries $\|\mathbf{B}\|_\infty < M$, where $\tau \in \mathbb{Z}_+$ and $M \in \mathbb{R}_+$.

Theorem 9.1: Suppose that the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of the linear control network (56) satisfies Assumption 3.1, its counting measure $\mu_{\mathcal{G}}$ enjoys the polynomial growth property (13), and the state matrix $\mathbf{B}$ is a strictly negative definite matrix in $\mathcal{B}_\tau(M) \cap \mathcal{B}^2$. A positive integer N_0 exists such that

$$\tilde{B}_{N_0} := \inf_{i_m \in \mathcal{V}_{N_0}} \tilde{B}_{N_0}(i_m)$$

$$\geq 4M \sqrt{\frac{\alpha_2^*}{\alpha_1^*}} D_1(\mathcal{G}) \tau(\tau+1)^d N_0^{-1} \quad (58)$$

in which $\tilde{B}_N(i)$ is defined by (52). For every pair of subsystems $k, l \in \mathcal{V}$ with $\rho(k,l) \leq \tau$, let us define the following quantities

$$\eta_{kl} = \inf_{\substack{\rho(k,i_m) \leq N_0 \\ \rho(l,i_m) \leq N_0}} \inf_{\|\mathbf{P}_{i_m}^{N_0} \mathbf{c}\|_2 = 1} \Re\left((\mathbf{P}_{i_m}^{N_0} \mathbf{c})^* \mathbf{E}_{kl} \, \mathbf{B} \, \mathbf{P}_{i_m}^{N_0} \mathbf{c} \right) \quad (59)$$

and

$$\beta_{kl} = \sup_{\substack{\rho(k,i_m) \leq N_0 \\ \rho(l,i_m) \leq N_0}} \sup_{\|\mathbf{P}_{i_m}^{N_0} \mathbf{c}\|_2 = 1} \Re\left((\mathbf{P}_{i_m}^{N_0} \mathbf{c})^* \mathbf{E}_{kl} \mathbf{B} \mathbf{P}_{i_m}^{N_0} \mathbf{c} \right), \quad (60)$$

where $\mathbf{P}_{i_m}^{N_0}$ is the projection matrix onto the eigenspace of the localized matrix $\chi_{i_m}^{N_0} \mathbf{B}^2 \chi_{i_m}^{N_0}$ corresponding to its smallest eigenvalue, which is equal to $\left(\tilde{B}_{N_0}(i_m) \right)^2$. Then, the following design rules hold:

(i) When $\eta_{kl} > 0$, there exists $\epsilon_0 > 0$ such that for all w in $(0, \epsilon_0)$ the resulting closed-loop network (57) is exponentially stable with the state matrix $\mathbf{B} + w\mathbf{E}_{kl}$ that still belongs to $\mathcal{B}_\tau(M)$.

(ii) When $\beta_{kl} < 0$, there exists $\epsilon_1 > 0$ such that for all w in $(-\epsilon_1, 0)$ the resulting closed-loop network (57) is exponentially stable with the state matrix $\mathbf{B} + w\mathbf{E}_{kl}$ that still belongs to $\mathcal{B}_\tau(M)$.

A detailed proof of the above theorem can be found in Appendix C. Suppose that $\{\mathbf{e}_{i_{m_1}}^{N_0}, \ldots, \mathbf{e}_{i_{m_k}}^{N_0}\}$ is an orthonormal basis of the eigenspace corresponding to the smallest eigenvalue of (53). Then, the projection matrix in Theorem 9.1 can be explicitly represented by

$$\mathbf{P}_{i_m}^{N_0} = \sum_{s=1}^{k} \mathbf{e}_{i_{m_s}}^{N_0} (\mathbf{e}_{i_{m_s}}^{N_0})^T.$$

When the smallest eigenvalue is simple with normalized eigenvector $\mathbf{q}_{i_m}^{N_0}$, the project matrix is given by

$$\mathbf{P}_{i_m}^{N_0} = \mathbf{q}_{i_m}^{N_0} (\mathbf{q}_{i_m}^{N_0})^*.$$

In this case, the norm constraint $\|\mathbf{P}_{i_m}^{N_0} \mathbf{c}\|_2 = 1$ implies that $\mathbf{P}_{i_m}^{N_0} \mathbf{c} = \mathbf{q}_{i_m}^{N_0}$. This gives us the following closed-form solutions

$$\inf_{\|\mathbf{P}_{i_m}^{N_0} \mathbf{c}\|_2 = 1} \Re\left((\mathbf{P}_{i_m}^{N_0} \mathbf{c})^* \mathbf{E}_{kl} \mathbf{B} \mathbf{P}_{i_m}^{N_0} \mathbf{c} \right) = \Re\left((\mathbf{q}_{i_m}^{N_0})^* \mathbf{E}_{kl} \mathbf{B} \, \mathbf{q}_{i_m}^{N_0} \right)$$
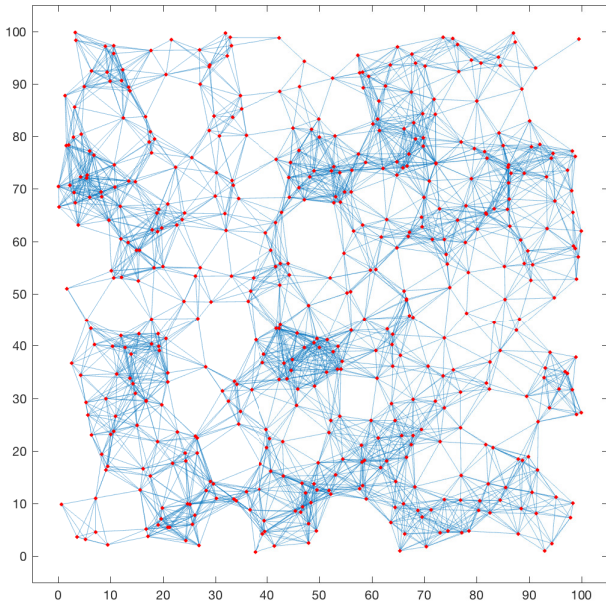
Fig. 4: The red dots show spatial locations of 500 subsystems that are randomly and uniformly distributed in a square-shape region of size 100×100. The blue links represent the communication graph among these subsystems as discussed in Section X.

and

$$\sup_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2 = 1} \Re\big((\mathbf{P}_{i_m}^{N_0}\mathbf{c})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{P}_{i_m}^{N_0}\mathbf{c}\big) = \Re\left((\mathbf{q}_{i_m}^{N_0})^*\mathbf{E}_{kl}\mathbf{B}\,\mathbf{q}_{i_m}^{N_0}\right)$$

that are useful in calculating quantities η_{kl} and β_{kl}. Computation of quantities η_{kl} and β_{kl} only involve those entries $b(i,j)$ of state matrix $\mathbf{B}$ whose indices satisfy

$$i, j \in B(k, 2N_0 + \tau) \cap B(l, 2N_0 + \tau).$$

Therefore, the requirements $\eta_{kl} > 0$ and $\beta_{kl} < 0$ in Theorem 9.1 can be verified by utilizing localized information about the state matrix $\mathbf{B}$ in neighborhoods of subsystems $k, l \in \mathcal{V}$.

The design parameter N_0 determines the size of neighborhoods required to compute quantities η_{kl} and β_{kl} using localized matrices. Suppose that N_0 and $\tilde{B}_{N_0}$ are chosen properly to satisfy the inequality (58). According to (4), (32), and (35), stability threshold of all networks with state matrices in $\mathcal{B}_\tau(M)$ is lower bounded by $\tilde{A}_{N_0} := \frac{1}{2}\tilde{B}_{N_0}\sqrt{\alpha_1^*/\alpha_2^*}$. Theorem 9.1 shows that state matrix of the resulting closed-loop networks still belong to $\mathcal{B}_\tau(M)$, which implies that their stability threshold is guaranteed to be greater than $\tilde{A}_{N_0}$.

In summary, the result of Theorem 9.1 asserts that: a properly chosen positive feedback gain w when $\eta_{kl} > 0$, and a properly chosen negative feedback gain w when $\beta_{kl} < 0$, will both result in exponentially stable closed-loop networks with guaranteed stability thresholds.

Remark 9.2: The conclusions of Theorem 9.1 will remain true if matrix $\mathbf{E}_{kl}$ is replaced by rotation matrix $\mathbf{R}_{kl}(\theta)$ whose (k,l)-th and (l,k)-th entries are $\sin\theta$, (k,k)-th entry is $\cos\theta$,
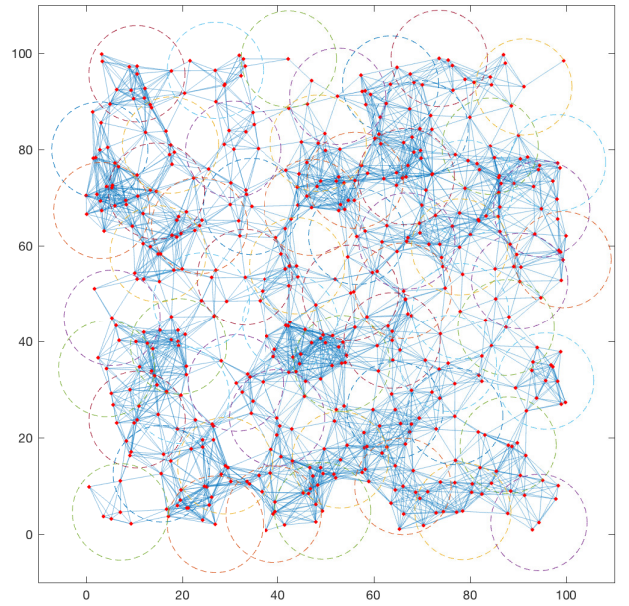

Fig. 5: A 2-covering of the communication graph in Figure 4.

and (l,l)-th entry is $-\cos\theta$ for some $0 \le \theta \le \pi$. Moreover, one can establish similar results when the matrix $\mathbf{E}_{kl}$ in Theorem 9.1 is replaced by $\mathbf{L}_{ij}$ where $\mathbf{L}_{ij} = \mathbf{e}_i\mathbf{e}_i^* + \mathbf{e}_j\mathbf{e}_j^* - \mathbf{E}_{ij}$ and $\mathbf{e}_i$'s are the standard basis for ℓ^2. This is particularly useful when the state matrix $\mathbf{B}$ is a graph Laplacian.

X. NUMERICAL SIMULATIONS

In this section, we interpret and illustrate some of the key concepts that are described and utilized in previous sections. We consider a linear dynamical network consisting of 500 subsystems which are randomly and uniformly distributed over a square-shape region of size 100×100 square meter. Let us denote spatial location of subsystem $i \in \{1, 2, \ldots, 500\}$ by $x_i \in [0, 100] \times [0, 100]$. The communication graph of this network is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and defined as follows: there is an undirected communication link between subsystems i and j, i.e., $\{i, j\} \in \mathcal{E}$, only if the Euclidean distance between subsystems i and j, i.e., $\|x_i - x_j\|_2$, is less than or equal to 10 meter; otherwise, there is no communication link between the two subsystems. Figure 4 depicts a sample communication graph obtained according to the above procedure, and Figure 5 is a 2-covering of the underlying connected graph, where the graph has diameter 18, Beurling dimension 2 and density 8.1389, and the number of leading subsystems is 62 (i.e., there are 62 localized regions).

We assume that state variable of each subsystem is scalar, i.e., $\psi_i \in \mathbb{R}$ for all $i \in \{1, 2, \ldots, 500\}$. We utilize shortest path on communication graph $\mathcal{G}$ as geodesic distance $\rho(i, j)$ to define state matrices of this class of linear networks. In our simulations, the bandwidth is set to $\tau = 1$. We construct state matrix $\mathbf{A} = [a(i,j)] \in \mathbb{R}^{500 \times 500}$ of our linear networks using
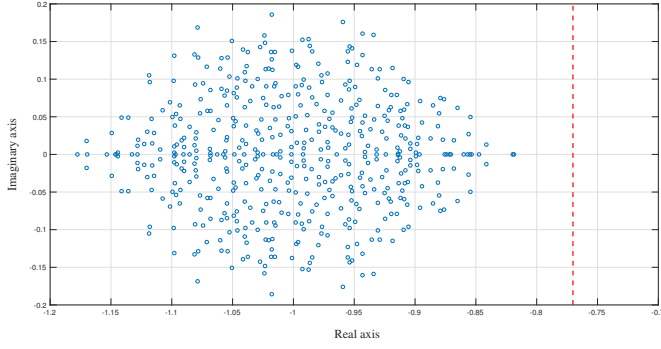
Fig. 6: Location of all 500 eigenvalues of the state matrix of the linear dynamical network as discussed in Section X.
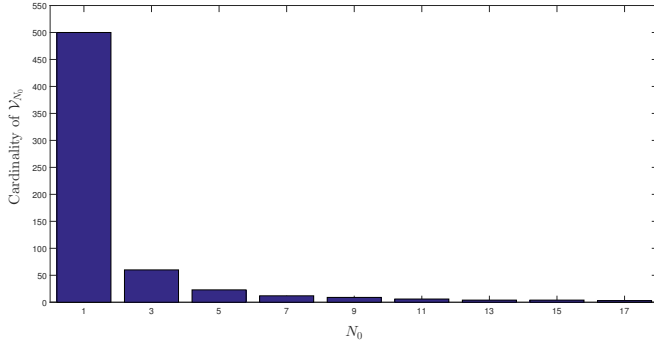

Fig. 8: The lower bound for B_{N_0} decreases as N_0 increases.

Fig. 7: Number of leading subsystems in N_0-covering decreases as N_0 increases.

the communication graph of Figure 4,

$$
a(i,j) = \begin{cases} -1 & \text{if } j = i \\ \text{sgn}(\zeta)\ 0.05\ e^{-\alpha\|x_i - x_j\|^\beta} & \text{if } 0 < \rho(j,i) \leq \tau \\ 0 & \text{if } \rho(j,i) > \tau, \end{cases}
$$

where decay parameters are $\alpha = 0.05$ and $\beta = 0.9$. In order to show that our methodology works for a broad class of systems, sign of each entry $a(i,j)$ is chosen randomly using $\text{sgn}(\zeta)$, where $\text{sgn}(\cdot)$ is the sign function and ζ is a random variable drawn from the standard normal distribution. After executing these steps, we adopted one sample matrix $\mathbf{A}$ for our simulation purposes. The resulting linear dynamical network is time-invariant, whose dynamics is governed by (5) with state vector $\psi \in \mathbb{R}^{500}$. The value of the Schur norm (3) of the state matrix is $\|\mathbf{A}\|_{\mathcal{S}} = 2.0057$ and all eigenvalues of $\mathbf{A}$ are located in the left-hand-side of the imaginary axis as it is shown in Figure 6. The red-dashed line in Figure 6 corresponds to $\{z \in \mathbb{C} \mid z = -A_0 + \xi\sqrt{-1} \text{ for all } \xi \in \mathbb{R}\}$, where the global stability threshold A_0 is equal to 0.7702.

In Figure 7, we show that the number of leading agents, according to Definition 4.2, in a N_0-covering of the network decreases as N_0 increases. We applied the algorithm described in Definition 4.2 in order to find N_0-coverings. This algorithm may not be optimal, but it works well for both finite and infinite
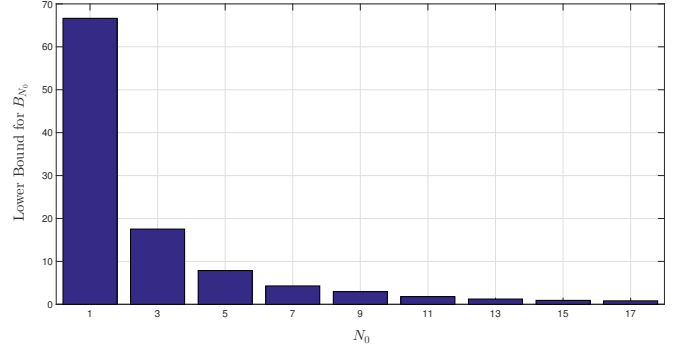
graphs. The following table shows our data for a sample execution of this algorithm on $\mathcal{G}$:

N_0	1	3	5	7	9	11	13	15	17	18
$\#\mathcal{V}_{N_0}$	500	55	23	14	9	7	6	3	3	1

We note that the number of leading subsystems is equal to the network size when $N_0 = 1$, and it is equal to 1 when N_0 is equal to the graph diameter.

The parameter B_{N_0} in the inequality (33) and (34) can be interpreted as an estimate for the global stability threshold A_0; cf. with inequalities (23) and (24). Figure 8 depicts that the quantity in the right hand side of the inequality (32), which is a lower bound for the localized stability threshold B_{N_0}, decays rapidly as radius of N_0-covering increases. The following table gives the corresponding B_{N_0}, which has been rounded up to fit in our table, for a sample simulation:

N_0	1	3	5	7	9	11	13	15	17	18
The lower bound for B_{N_0}	66.6	16.7	7.7	4.3	2.7	1.9	1.5	0.92	0.81	0.45

We observe that for $N_0 = 18$, the lower bound for B_{N_0} is around 0.45. This is compatible with the estimate in (40) as 0.45 is a lower bound for the localized stability threshold.

The inequality (35) shows the relationship between the global stability threshold A_0 and localized stability threshold B_{N_0} and how one should choose a proper value for N_0 in Theorem 7.1. Figure 9 illustrates how the lower bound in inequality (35) decays as N_0 increases. The red-dashed line marks the value of $A_0 = 0.7702$. In computing this lower bound, the value of B_{N_0} is assumed to be equal to the lower bound of (32). According to Figure 9 and the following table, N_0 should be chosen any integer number between 5 and 18.

N_0	1	3	5	6	7	9	11	13	15	17	18
A_0	4.01	1.34	0.80	0.67	0.57	0.45	0.37	0.31	0.27	0.24	0.22

In Theorem 7.1, parameters α_1^* and α_2^* play a crucial role in having a proper estimate of the localized stability threshold B_{N_0}. Figure 10 depicts quantitative behavior of these two parameters. The numerical values for α_2^* is given in the following table.
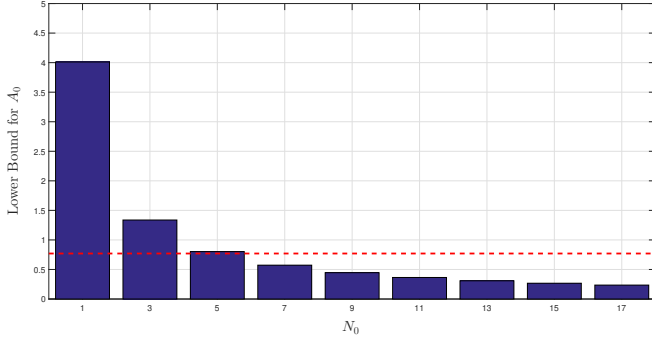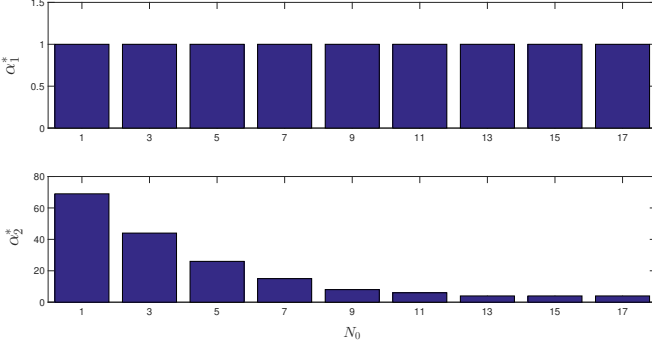
Fig. 9: Lower bound for A_0

Fig. 10: Behavior of α_1^* and α_2^* as size of covering N_0 increases.

N_0	1	3	5	7	9	11	13	15	17	18
α_2^*	69	43	25	12	9	7	5	3	3	1

linear systems similar to (5). Whenever the resulting state matrix is banded for some small $\tau > 0$, our proposed methodology can be applied to verify stability of the linear PDEs using spatially localized certificates.

Stability of Spatially Distributed Nonlinear Systems: Our methodology can be extended to check stability of equilibria of spatially distributed systems with nonlinear dynamics. Let us consider a nonlinear system of the form

$$\frac{d}{dt}\psi = F(\psi), \quad t \ge 0, \tag{61}$$

where $\psi(t) = [\psi_i(t)]_{i \in \mathcal{V}} \in \ell^2$ and $F : \ell^2 \to \ell^2$. It is assumed that a suitable notion of a solution exists for this system. Following Definition 3.3, we say that the nonlinear system (61) is τ-banded over the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $F(\psi) = [F_i(\Psi_i)]_{i \in \mathcal{V}}$ for some continuously differentiable functions F_i on $\mathbb{C}^q$, where $q = \#(B(i,\tau) \cap \mathcal{V})$ and $\Psi_i = [\psi_j]_{j \in B(i,\tau) \cap \mathcal{V}}$ for all $i \in \mathcal{V}$. Let us define the gradient $\mathbf{A}(\psi^*) = \nabla_\psi F(\psi^*)$ (with respect to ψ at working point ψ^*) of the nonlinear system (61) with (i,j)-th entry $a(i,j)$ given by

$$a(i,j) = \begin{cases} \frac{\partial F_i}{\partial \psi_j}(\psi^*) & \text{if} \quad \rho(i,j) \le \tau \\ 0 & \text{if} \quad \rho(i,j) > \tau, \end{cases}$$

where ρ is a geodesic distance on $\mathcal{G}$. By assuming that ψ^* is a hyperbolic equilibrium and the initial state $\psi(0)$ is close to ψ^*, our proposed localized stability certificates can be applied to the nonlinear dynamic system (61) with the linearized state matrix $\mathbf{A}(\psi^*)$ to infer stability in a spatially localized manner.

Stability of Spatially Distributed Systems on ℓ^∞: Let us consider the following control system

$$\frac{d}{dt}\psi(t) = \mathbf{A}\psi(t) + \boldsymbol{\xi}(t), \tag{62}$$

with initial condition $\psi(0) \in \ell^\infty$, that is driven by a time-dependent bounded control or exogenous noise $\boldsymbol{\xi}(t) = [\xi_i(t)]_{i \in \mathcal{V}}$. Suppose that the control system is exponentially stable on ℓ^∞, i.e., there exist strictly positive constants C and α such that

$$\|e^{\mathbf{A}t}\psi(0)\|_\infty \le Ce^{-\alpha t}\|\psi(0)\|_\infty \tag{63}$$

for all $t \ge 0$, then we have

$$
\begin{aligned}
\|\psi(t)\|_\infty &\le \left\| \int_0^t e^{\mathbf{A}(t-s)}\boldsymbol{\xi}(s)ds + e^{\mathbf{A}t}\psi(0) \right\|_\infty \\
&\le Ce^{-\alpha t}\|\psi(0)\|_\infty + C\int_0^t e^{-\alpha(t-s)}\|\boldsymbol{\xi}(s)\|_\infty ds \\
&\le Ce^{-\alpha t}\|\psi(0)\|_\infty + C\alpha^{-1} \sup_{0 \le t \le s} \|\boldsymbol{\xi}(s)\|_\infty.
\end{aligned}
$$

This implies that control system (62) with bounded input has

XI. DISCUSSION AND CONCLUSION

This work proposes a decentralized algorithm to verify exponential stability of linear dynamical network that are defined over spatial proximity communication graphs. Several necessary and sufficient conditions have been formulated that can be utilized to check exponential stability of a large class of finite- and infinite-dimensional linear systems. There are several related problems and areas that can benefit from our proposed methodology in this paper:

Stability of Partial Differential Equations (PDE): There has been continued interested in characterizing stability conditions of dynamical systems with PDE models [2], where the main focus has been on obtaining centralized conditions. Stability of linear PDEs with spatially-invariant coefficients can be characterized in terms of spectrum of the corresponding linear operators [2], [3], where similar spectral methods cannot be applied to check stability of linear PDEs with spatially varying coefficients[2]. Some linear PDEs with spatially invariant or varying coefficients can be discretized over their underlying spatial domain at certain scale and they be approximated by the resulting counterparts, which results in infinite-dimensional

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[2]This can be compared to the more familiar case of finite-dimensional linear time-invariant (LTI) systems versus linear time-varying systems (LTV). Stability of an LTI system can be inferred from location of eigenvalues of its state matrix, while stability of a LTV system cannot be deduced from eigenvalues of its time-varying state matrix.

bounded state $\psi(t)$ for all $t \geq 0$. It is proven in [29] that if the linear system (62) is exponentially stable on ℓ^2, i.e., the inequality (6) holds, then it is also exponential stable on ℓ^∞, i.e., the inequality (63) holds. When the state matrix $\mathbf{A}$ belongs to $\mathcal{B}_\tau \cap \mathcal{B}^2$, we have that the constant C in (63) will depend only on the constants E, α given in (6), Beurling dimension d, Beurling density $D_1(\mathcal{G})$ and doubling constant $D_0(\mathcal{G})$ of the graph $\mathcal{G}$, bandwidth τ, and the value of $\|\mathbf{A}\|_\infty$. From this argument, we conclude that our proposed methodology in this paper can be applied to the control system (62) driven by input (which can be a feedback control law or exogenous noise) to infer global stability in a decentralized manner.

APPENDIX
PROOF OF THEOREMS

A. Proof of Theorem 5.1

We divide the proof into the following three implications

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longleftrightarrow \text{(iii)} \Longleftrightarrow \text{(iv)}$$

and prove them one by one. First, we start with (i)$\Longleftrightarrow$(ii). The sufficiency follows as the quantity $\|(z\mathbf{I} - \mathbf{A})^{-1}\|_{\mathcal{B}^2}$ is continuous about z with $\Re(z) \geq 0$ and it tends to zero as $|z| \to +\infty$. For the necessity, we have

$$(w\mathbf{I} - \mathbf{A})^{-1} = ((w-a)\mathbf{I} - \mathbf{A})^{-1} \sum_{n=0}^{\infty} (-a((w-a)\mathbf{I} - \mathbf{A})^{-1})^n \tag{64}$$

for all $w \in \mathbb{C}$ with $\Re(w) > -A_0$, where $a = \min\{0, \Re(w)\}$. The Neumann series expansion in (64) holds as

$$|a|\|((w-a)\mathbf{I} - \mathbf{A})^{-1}\|_{\mathcal{B}^2} \leq |a|/A_0 < 1.$$

Therefore,

$$\sigma(\mathbf{A}) \subset \{z \in \mathbb{C}, \; \Re(z) \leq -A_0\}, \tag{65}$$

which proves statement (i) with $\delta = A_0$.

Next, we show (ii)$\Longleftrightarrow$(iii). The sufficiency holds as matrices $z\mathbf{I} - \mathbf{A}$ and $z\mathbf{I} - \mathbf{A}^*$, where $\Re(z) \geq 0$, have uniformly bounded inverses. To prove the necessity, let us pick a $z \in \mathbb{C}$ with $\Re(z) \geq 0$. By the ℓ^2-stability property (16), it suffices to prove that the range of $z\mathbf{I} - \mathbf{A}$ is the entire ℓ^2 space. Let us suppose, on the contrary, that orthogonal complement of the range is nontrivial, i.e., there exists $\mathbf{0} \neq \mathbf{d} \in \ell^2$ such that $\mathbf{d}^*(z\mathbf{I} - \mathbf{A})\mathbf{c} = 0$ for all $\mathbf{c} \in \ell^2$. Thus, $(\bar{z}\mathbf{I} - \mathbf{A}^*)\mathbf{d} = \mathbf{0}$, which together with the ℓ^2-stability property (16) for $\bar{z}\mathbf{I} - \mathbf{A}^*$ implies that $\mathbf{d} = \mathbf{0}$. This is a contradiction, which proves our claim.

Finally, we prove (iii)$\Longleftrightarrow$(iv). The implication follows from the following:

$$\inf_{\Re(z) \geq 0} \|(z\mathbf{I} - \mathbf{A})\mathbf{c}\|_2^2$$
$$= \inf_{\Re(z) \geq 0} |z|^2 \mathbf{c}^*\mathbf{c} - \bar{z}\mathbf{c}^*\mathbf{A}^*\mathbf{c} - z\mathbf{c}^*\mathbf{A}\mathbf{c} + \mathbf{c}^*\mathbf{A}^*\mathbf{A}\mathbf{c}$$
$$= \mathbf{c}^*\mathbf{A}^*\mathbf{A}\mathbf{c} - \frac{|\mathbf{c}^*\mathbf{A}_{ah}\mathbf{c}|^2 + (\max(0, \mathbf{c}^*\mathbf{A}_h\mathbf{c}))^2}{\|\mathbf{c}\|_2^2}$$

and

$$\inf_{\Re(z) \geq 0} \|(z\mathbf{I} - \mathbf{A}^*)\mathbf{c}\|_2^2$$
$$= \mathbf{c}^*\mathbf{A}\mathbf{A}^*\mathbf{c} - \frac{|\mathbf{c}^*\mathbf{A}_{ah}\mathbf{c}|^2 + (\max(0, \mathbf{c}^*\mathbf{A}_h\mathbf{c}))^2}{\|\mathbf{c}\|_2^2}$$

for all $\mathbf{c} \in \ell^2$.

B. Proof of Theorem 7.1

Let us pick $z \in \mathbb{C}^+$ and $\mathbf{c} \in \ell^2$. By Theorem 5.1, it suffices to prove the uniform stability for matrices $z\mathbf{I} - \mathbf{A}$ and $z\mathbf{I} - \mathbf{A}^*$, i.e.,

$$\|(z\mathbf{I} - \mathbf{A})\mathbf{c}\|_2 \geq \frac{B_{N_0}}{2} \sqrt{\frac{\alpha_1}{\alpha_2}} \|\mathbf{c}\|_2 \tag{66}$$

and

$$\|(z\mathbf{I} - \mathbf{A}^*)\mathbf{c}\|_2 \geq \frac{B_{N_0}}{2} \sqrt{\frac{\alpha_1}{\alpha_2}} \|\mathbf{c}\|_2. \tag{67}$$

For every subsystem $i \in \mathcal{V}$ and integer $N \geq 0$, we define operator $\Psi_i^N : \ell^2 \to \ell^2$ by

$$\Psi_i^N : \; [c(j)]_{j \in \mathcal{V}} \longmapsto [\psi_0(\rho(i,j)/N)c(j)]_{j \in \mathcal{V}}$$

in which ψ_0 is the trapezoid function given by

$$\psi_0(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 2 - 2|t| & \text{if } 1/2 < |t| \leq 1 \\ 0 & \text{if } |t| > 1. \end{cases}$$

Similar to the truncation operator χ_i^N in (22), the operator Ψ_i^N localizes a vector to the N-neighborhood of subsystem i, which can be represented as a diagonal matrix whose diagonal entries are $\psi_0(\rho(i,j)/N)$ for all $j \in \mathcal{V}$, and it can be thought of as a smooth version of the truncation operator $\chi_i^N : \ell^2 \to \ell^2$ in (22). By the local stability assumption (33), we have

$$\|(z\mathbf{I} - \mathbf{A})\Psi_{i_m}^{N_0}\mathbf{c}\|_2 \geq B_{N_0}\|\Psi_{i_m}^{N_0}\mathbf{c}\|_2 \tag{68}$$

for all $\mathbf{c} \in \ell^2$ and $i_m \in \mathcal{V}_{N_0}$. Let us denote $\mathbf{A} = [a(i,j)]_{i,j \in \mathcal{V}}$. Then, for every $\mathbf{c} = [c(j)]_{j \in \mathcal{V}}$ we obtain

$$\sum_{i_m \in \mathcal{V}_{N_0}} \|(\mathbf{A}\Psi_{i_m}^{N_0} - \Psi_{i_m}^{N_0}\mathbf{A})\mathbf{c}\|_2^2$$
$$= \sum_{i_m \in \mathcal{V}_{N_0}} \|(\mathbf{A}\Psi_{i_m}^{N_0} - \Psi_{i_m}^{N_0}\mathbf{A})\chi_{i_m}^{2N_0}\mathbf{c}\|_2^2$$
$$\leq \sum_{i_m \in \mathcal{V}_{N_0}} \sum_{i \in \mathcal{V}} \left\{ \sum_{\rho(i,j) \leq \tau} |a(i,j)|\, \chi_{B(i_m, 2N_0)}(j)\, |c(j)| \right.$$
$$\left. \times \left| \psi_0\left(\frac{\rho(i,i_m)}{N_0}\right) - \psi_0\left(\frac{\rho(j,i_m)}{N_0}\right) \right| \right\}^2$$
$$\leq 4 \sum_{i_m \in \mathcal{V}_{N_0}} \sum_{i \in \mathcal{V}} \left\{ \sum_{\rho(i,j) \leq \tau} |a(i,j)|\, \frac{\rho(i,j)}{N_0}\, \chi_{B(i_m, 2N_0)}(j)\, |c(j)| \right\}^2$$
$$\leq 4\tau^2 N_0^{-2} \|\mathbf{A}\|_{\mathcal{S}}^2 \sum_{i_m \in \mathcal{V}_{N_0}} \|\chi_{i_m}^{2N_0}\mathbf{c}\|_2^2$$
$$\leq 4\tau^2 N_0^{-2} \alpha_2 \|\mathbf{A}\|_{\mathcal{S}}^2 \|\mathbf{c}\|_2^2, \tag{69}$$

in which the second inequality follows from the Lipschitz property for the trapezoid function ψ_0 and the third one holds by the second inequality in (4). By combining (68) and (69), we get

$$
\sqrt{\alpha_2}\,\|(z\mathbf{I}-\mathbf{A})\mathbf{c}\|_2 \;\geq\; \Big(\sum_{i_m\in\mathcal{V}_{N_0}}\|\Psi_{i_m}^{N_0}(z\mathbf{I}-\mathbf{A})\mathbf{c}\|_2^2\Big)^{1/2}
$$

$$
\geq\Big(\sum_{i_m\in\mathcal{V}_{N_0}}\|(z\mathbf{I}-\mathbf{A})\Psi_{i_m}^{N_0}\mathbf{c}\|_2^2\Big)^{1/2}
$$

$$
-\Big(\sum_{i_m\in\mathcal{V}_{N_0}}\|(\mathbf{A}\Psi_{i_m}^{N_0}-\Psi_{i_m}^{N_0}\mathbf{A})\mathbf{c}\|_2^2\Big)^{1/2}
$$

$$
\geq B_{N_0}\Big(\sum_{i_m\in\mathcal{V}_{N_0}}\|\Psi_{i_m}^{N_0}\mathbf{c}\|_2^2\Big)^{1/2}-2\sqrt{\alpha_2}\,\tau\,N_0^{-1}\,\|\mathbf{A}\|_{\mathcal{S}}\|\mathbf{c}\|_2
$$

$$
\geq\Big(B_{N_0}\sqrt{\alpha_1}-2\sqrt{\alpha_2}\,\tau\,N_0^{-1}\|\mathbf{A}\|_{\mathcal{S}}\Big)\,\|\mathbf{c}\|_2.
$$

This together with (32) proves (66). By applying similar arguments, we can establish the lower bound estimate in (67).

C. Proof of Theorem 9.1

According to our assumptions, the state matrix $\mathbf{B}$ is a strictly negative definite matrix in $\mathcal{B}_\tau(M)\cap\mathcal{B}^2$. By Theorem 8.4, it suffices to find proper positive or negative weight adjustment w such that

$$
\mathbf{c}^*\chi_{i_m}^{N_0}\big(\mathbf{B}+w\mathbf{E}_{kl}\big)\chi_{i_m}^{N_0}\mathbf{c}\leq 0 \tag{70}
$$

and

$$
\big\|(\mathbf{B}+w\mathbf{E}_{kl})\chi_{i_m}^{N_0}\mathbf{c}\big\|_2^2 \;\geq\; \big(\tilde{B}_{N_0}\big)^2\,\big\|\chi_{i_m}^{N_0}\mathbf{c}\big\|_2^2 \tag{71}
$$

hold for all $i_m\in\mathcal{V}_{N_0}$ and $\mathbf{c}\in\ell^2$, where $\tilde{B}_{N_0}$ is the constant in (50) for the matrix $\mathbf{B}$.

The proof of part (i) of the theorem is as follows. Since $\mathbf{B}\in\mathcal{B}_\tau(M)$ and $\rho(k,l)\leq\tau$, we have $|b(k,l)|<M$. Hence,

$$
\mathbf{B}+w\,\mathbf{E}_{kl}\in\mathcal{B}_\tau(M)
$$

if

$$
|w|\leq M-|b(k,l)|. \tag{72}
$$

We observe that (32) holds according to (4), (58), and the assumption $\mathbf{B}\in\mathcal{B}_\tau(M)$. Therefore, $\mathbf{B}$ is strictly negative definite by Theorem 7.1. Moreover, it follows from (66) and (58) that

$$
\mathbf{c}^*\mathbf{B}\mathbf{c} \;\leq\; -\frac{B_{N_0}}{2}\sqrt{\frac{C_0}{C_1}}\,\|\mathbf{c}\|_2^2
$$

$$
\leq\; -2MD_1(\mathcal{G})\tau(\tau+1)^d\,N_0^{-1}\,\|\mathbf{c}\|_2^2 \tag{73}
$$

for all $\mathbf{c}\in\ell^2$. Direct calculations reveal that

$$
\big|\mathbf{c}^*\mathbf{E}_{kl}\mathbf{c}\big|\;\leq\;\|\mathbf{c}\|_2^2\quad\text{for all}\quad\mathbf{c}\in\ell^2.
$$

This together with (73) implies that $\mathbf{B}+w\mathbf{E}_{kl}$ is negative definite matrices and, as a result, inequality (70) holds for all

w satisfying

$$
|w|<2MD_1(\mathcal{G})\,\tau(\tau+1)^d\,N_0^{-1}. \tag{74}
$$

Now, suppose that $i_m\in\mathcal{V}_{N_0}$ such that $k,l\in B(i_m,N_0)$, where k and l are indices of $\mathbf{E}_{kl}$. From the definition of the projection matrix $\mathbf{P}_{i_m}^{N_0}$, it follows that there exists $\tilde{C}_{N_0}(i_m)>B_{N_0}(i_m)$ such that

$$
\|\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \;\geq\; \tilde{B}_{N_0}(i_m)^2\,\|\mathbf{P}_{i_m}\mathbf{c}\|_2^2
$$

$$
+\,\tilde{C}_{N_0}(i_m)^2\,\|\chi_{i_m}^{N_0}\mathbf{c}-\mathbf{P}_{i_m}\mathbf{c}\|_2^2 \tag{75}
$$

for all $\mathbf{c}\in\ell^2$. In fact, the second smallest eigenvalue of the matrix $\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}$, if it exists, can be employed as the constant $\tilde{C}_{N_0}(i_m)$ in (75). Let us choose $\mathbf{c}_1=\mathbf{P}_{i_m}\mathbf{c}$ and $\mathbf{c}_2=\chi_{i_m}^{N_0}\mathbf{c}-\mathbf{P}_{i_m}\mathbf{c}$ for $\mathbf{c}\in\ell^2$. Then,

$$
\|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \;=\; \|\mathbf{c}_1\|_2^2 \;+\; \|\mathbf{c}_2\|_2^2.
$$

For any positive weight w, we obtain

$$
\|(\mathbf{B}+w\mathbf{E}_{kl})\chi_{i_m}^{N_0}\mathbf{c}\|_2^2
$$

$$
=\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}\mathbf{c}+2w\,\Re\big(\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{E}_{kl}\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\big)+w^2\|\mathbf{E}_{kl}\chi_{i_m}^{N_0}\mathbf{c}\|_2^2
$$

$$
\geq\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}\mathbf{c}+2w\,\Re\big(\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{E}_{kl}\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\big)
$$

$$
\geq\tilde{B}_{N_0}(i_m)^2\,\|\mathbf{c}_1\|_2^2\;+\;\tilde{C}_{N_0}(i_m)^2\,\|\mathbf{c}_2\|_2^2\;+\;2w\,\Re\big(\mathbf{c}_1^*\mathbf{E}_{kl}\mathbf{B}\mathbf{c}_1\big)
$$

$$
-\,2w\,\|\mathbf{E}_{kl}\mathbf{c}_2\|_2\|\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\|_2\;-\;2w\,\|\mathbf{E}_{kl}\mathbf{c}_1\|_2\,\|\mathbf{B}\mathbf{c}_2\|_2
$$

$$
\geq\Big(\tilde{B}_{N_0}(i_m)^2\;+\;2w\eta_{kl}\Big)\,\|\mathbf{c}_1\|_2^2
$$

$$
+\Big(\tilde{C}_{N_0}(i_m)^2\;-\;2wMD_1(\mathcal{G})(\sigma+1)^d\Big)\,\|\mathbf{c}_2\|_2
$$

$$
-\,4wMD_1(\mathcal{G})(\sigma+1)^d\,\|\mathbf{c}_1\|_2\,\|\mathbf{c}_2\|_2
$$

$$
\geq\tilde{B}_{N_0}(i_m)^2\,\|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2+\Big(\tilde{C}_{N_0}(i_m)^2\;-\;\tilde{B}_{N_0}(i_m)^2\Big)\,\|\mathbf{c}_2\|_2^2
$$

$$
-\,2wMD_1(\mathcal{G})(\sigma+1)^d\,\Big(\frac{D_1(\mathcal{G})(\sigma+1)^d}{\eta_{ij}}+1\Big)\,\|\mathbf{c}_2\|_2^2,
$$

in which the second inequality holds according to (75) and the third inequality follows from (4) and the observation

$$
\|\mathbf{E}_{kl}\mathbf{c}\|_2\leq\|\mathbf{c}\|_2\quad\text{for all }\mathbf{c}\in\ell^2.
$$

Hence, the inequality (71) holds when

$$
0<w<\frac{\eta_{kl}\big(\tilde{C}_{N_0}(i_m)^2\;-\;\tilde{B}_{N_0}(i_m)^2\big)}{2MD_1(\mathcal{G})(\sigma+1)^d\big(MD_1(\mathcal{G})(\sigma+1)^d+\eta_{kl}\big)}. \tag{76}
$$

This together with (72) and (74) proves the first conclusion with ϵ_0 given by

$$
\epsilon_0 \;=\; \min\Big\{M-|b(k,l)|,\;2MD_1(\mathcal{G})\sigma(\sigma+1)^dN_0^{-1},
$$

$$
\frac{\eta_{ij}\big(\tilde{C}_{N_0}(i_m)^2\;-\;\tilde{B}_{N_0}(i_m)^2\big)}{2MD_1(\mathcal{G})(\sigma+1)^d\big(MD_1(\mathcal{G})(\sigma+1)^d+\eta_{ij}\big)}\Big\}.
$$

The proof of part (ii) of the theorem is as follows. Similar to the arguments used in proof of part (i), for negative weight w, one can obtain

$$
\|(\mathbf{B}+w\mathbf{E}_{kl})\chi_{i_m}^{N_0}\mathbf{c}\|_2^2\geq\big(\tilde{B}_{N_0}(i_m)^2\;+\;2w\beta_{kl}\big)\,\|\mathbf{c}_1\|_2^2
$$

$$+ \left(\tilde{C}_{N_0}(i_m)^2 + 2wMD_1(\mathcal{G})(\sigma + 1)^d \right) \|\mathbf{c}_2\|_2$$
$$+ 4wMD_1(\mathcal{G})(\sigma + 1)^d \|\mathbf{c}_1\|_2 \|\mathbf{c}_2\|_2$$
$$\geq \tilde{B}_{N_0}(i_m)^2 \|\mathbf{c}_1\|_2^2 + \tilde{C}_{N_0}(i_m)^2 \|\mathbf{c}_2\|_2^2$$
$$+ 2wMD_1(\mathcal{G})(\sigma + 1)^d \left(1 - \frac{D_1(\mathcal{G})(\sigma + 1)^d}{\beta_{kl}} \right) \|\mathbf{c}_2\|_2^2.$$

Therefore, the second conclusion holds by letting

$$\epsilon_1 = \min \left\{ M - |b(k,l)|, 2MD_1(\mathcal{G})\sigma(\sigma + 1)^d N_0^{-1}, \right.$$
$$\left. \frac{-\beta_{ij}\left((\tilde{C}_{N_0}(i_m))^2 - (\tilde{B}_{N_0}(i_m))^2\right)}{2MD_1(\mathcal{G})(\sigma + 1)^d\left(MD_1(\mathcal{G})(\sigma + 1)^d - \beta_{ij}\right)} \right\}.$$

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Nader Motee (S'99-M'08-SM'13) received his B.Sc. degree in Electrical Engineering from Sharif University of Technology in 2000, M.Sc. and Ph.D. degrees from University of Pennsylvania in Electrical and Systems Engineering in 2006 and 2007 respectively. From 2008 to 2011, he was a postdoctoral scholar in the Control and Dynamical Systems Department at Caltech. He is currently an Associate Professor in the Department of Mechanical Engineering and Mechanics at Lehigh University. His current research area is distributed dynamical and control systems with particular focus on issues related to sparsity, performance, and robustness. He is a past recipient of several awards including the 2008 AACC Hugo Schuck best paper award, the 2007 ACC best student paper award, the 2008 Joseph and Rosaline Wolf best thesis award, a 2013 Air Force Office of Scientific Research Young Investigator Program award (AFOSR YIP), a 2015 NSF Faculty Early Career Development (CAREER) award, and a 2016 Office of Naval Research Young Investigator Program award (ONR YIP).

Qiyu Sun received the Ph.D. degree in Mathematics from Hangzhou University, Hangzhou, China, in 1990. He is currently a Professor of Mathematics at the University of Central Florida, Orlando, FL, USA. His prior position was with the Zhejiang University, China; the National University of Singapore, Singapore; Vanderbilt University, Nashville, TN, USA; and the University of Houston, Houston, TX, USA. His research interests include applied and computational harmonic analysis, sampling theory, phase retrieval and graph signal processing. He has published more than 100 papers and written a book An Introduction to Multiband Wavelets with N. Bi and D. Huang.