

# Fundamental Limits on Robustness Measures in Networks of Interconnected Systems\*

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**Abstract**—We investigate robustness of interconnected dynamical networks with respect to external distributed stochastic disturbances. In this paper, we consider networks with linear time-invariant dynamics. The  $\mathcal{H}_2$  norm of the underlying system is considered as a robustness index to measure the expected steady-state dispersion of the state of the entire network. We present new tight bounds for the robustness measure for general linear dynamical networks. We, then, focus on two specific classes of networks: first- and second-order consensus in dynamical networks. A weighted version of the  $\mathcal{H}_2$  norm of the system, so called  $L_Q$ -energy of the network, is introduced as a robustness measure. It turns out that when  $L_Q$  is the Laplacian matrix of a complete graph,  $L_Q$ -energy reduces to the expected steady-state dispersion of the state of the entire network. We quantify several graph-dependent and graph-independent fundamental limits on the  $L_Q$ -energy of the networks. Our theoretical results have been applied to two application areas. First, we show that in power networks the concept of  $L_Q$ -energy can be interpreted as the total resistive losses in the network and that it does not depend on specific structure of the underlying graph of the network. Second, we consider formation control with second-order dynamics and show that the  $L_Q$ -energy of the network is graph-dependent and corresponds to the energy of the flock.

## I. INTRODUCTION

The challenge of designing robust networks of interconnected systems lies at the core of theory of networked systems (see [1]–[7] and references in there). There are several design factors to be considered when designing a robust dynamical network, namely, individual system or link failures which result in a change in topology of the underlying graph of the network, as well as environmental and communication uncertainties which can be modeled as external network-wide stochastic perturbations and disturbances. Therefore, one of fundamental challenges is to investigate robustness properties of dynamical networks with respect to underlying graph structure as well as external distributed stochastic disturbances.

There have been several recent works on the robustness of first- and second-order consensus in dynamical networks [2]–[8]. In these works, the  $\mathcal{H}_2$  norm of the underlying system has been adopted as a robustness measure with respect to external stochastic disturbances. This robustness measure depends on the output (or performance measure) of the network and can result in various forms of input-output  $\mathcal{H}_2$  norms for a given network. In the existing literature,

including the above cited papers, it is common to define a specific output structure for the network and to perform the robustness analysis with respect to that given output. In order to propose a unified approach to handle general cases without explicitly defining the output of the network, we introduce the notion of  $L_Q$ -energy of a dynamical network which is a weighted version of  $\mathcal{H}_2$  norm of the system. Suppose that matrix  $C$  defines the structure of the output of the network. One can associate a weighted graph  $\mathcal{Q}$  to the positive semidefinite matrix  $L_Q = C^T C$ . In this setting, the structure of the output is implicitly incorporated into the  $L_Q$ -energy of the dynamical network. In this paper, we show that several existing robustness measures in the literature are special forms of our proposed robustness measure. We also derive several interesting results on how  $L_Q$ -energy of a dynamical network depends on the topology of the underlying graph of the network.

In Section IV, we obtain new tight bounds on the  $\mathcal{H}_2$  norm of general interconnected linear dynamical networks in terms of the eigenvalues of the state matrix of the system. This result enables us to characterize fundamental limits on the proposed robustness measure for several interesting dynamical networks and applications. In Section V, we apply the results of Section IV to study robustness properties of cyclic interconnected dynamical networks. This class of networks usually arise in modeling biological networks such as Glycolysis pathway [4], [5]. In Sections VI and VIII, we focus on two specific classes of dynamical networks: consensus in networks with first- and second-order dynamics. We quantify several graph-dependent and graph-independent fundamental limits on the  $L_Q$ -energy of the networks in Sections VII and IX. Our theoretical results have been applied to two applications areas. First, we show that in power networks the concept of  $L_Q$ -energy can be interpreted as the total resistive losses in the network and that it does not depend on specific structure of the underlying graph of the network. In [9], it is shown that the  $\mathcal{H}_2$  norm of the system can be computed for special class of networks whose matrices can be diagonalized simultaneously. In Section IX, we present general results and show that  $L_Q$ -energy scales with the product of the network size and the weighted mean of the ratios of line resistances to their reactances, which are not necessarily equivalent for all edges (i.e., transmission lines). At the end, we consider formation control with second-order dynamics and show that the  $L_Q$ -energy of the network is graph-dependent and corresponds to the energy of the flock (cf. [10]).

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## II. MATHEMATICAL PRELIMINARIES

In this section, some definitions and basic concepts, which are useful in the analyses of the paper, are brought. In this paper all of graphs that we consider are finite, simple and undirected. Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$  be a graph with vertex set  $\mathcal{V}(\mathcal{G})$ , edge set  $\mathcal{E}(\mathcal{G})$  and  $\mathcal{W}(\mathcal{G}) = \{w_{u,v} \in \mathbb{R}_+ | (u,v) \in \mathcal{E}(\mathcal{G})\}$  is a set of weights assigned to each edge of the graph (here  $\mathbb{R}_+$  denotes the set of nonnegative real numbers). Let  $n = |\mathcal{V}(\mathcal{G})|$  be the number of nodes in  $\mathcal{G}$  (also called the order of  $\mathcal{G}$ ) and  $m = |\mathcal{E}(\mathcal{G})|$  be the number of edges in  $\mathcal{G}$ . We define the adjacency matrix  $A = [a_{ij}]$  of  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ , such that  $a_{ij} = w_{i,j}$ . The Laplacian matrix of  $\mathcal{G}$  is defined as  $L = \Delta - A$ , where  $\Delta = [d_{ii}] = d(i) := \sum_j a_{ij}$  is a diagonal matrix with the node degrees on the diagonal. The eigenvalues of  $L$ , denoted by  $\lambda_1 = 0 \leq \lambda_2 \leq \dots \leq \lambda_n$ . And Similarly for  $1 \leq i \leq n$ , let  $u_i$  be the corresponding eigenvector of  $\lambda_i$ . The matrix of eigenvectors  $U = [u_1, u_2, \dots, u_n]$  represents an orthonormal basis for an  $n$ -dimensional Euclidean space. Since we have  $L = U\Lambda U^T$  where  $\Lambda$  is the diagonal matrix  $\Lambda = [\lambda_{ii}] = \lambda_i$ . The Laplacian matrix  $L$  of a graph is must naturally defined by the quadratic form it induces. We define the  $L$ -norm of a vector  $x \in \mathbb{R}^n$  as follows

$$\|x\|_L^2 = x^T L x = \sum_{(u,v) \in \mathcal{E}} \omega_{u,v} (x_u - x_v)^2. \quad (1)$$

Thus  $L$ -norm provides a measure of the smoothness of  $x$  over the edges in  $\mathcal{G}$ . The matrix  $L^+$  denotes the Moore-Penrose pseudo-inverse of  $L$ . Note that  $L^+$  is square, symmetric, doubly-centered and positive semi-definite [11].

Note that for a unweighted graph where  $w_{i,j} \in \{0, 1\}$ ,  $A$  and  $L$  are simply the standard adjacency matrix and Laplacian matrix of the graph  $\mathcal{G}$ , respectively. The  $n$ -vertex complete graph denoted by  $\mathcal{K}_n$  is the graph with the maximum number of edges. The  $n$ -vertex star, denoted by  $\mathcal{S}_n$  is the  $n$ -vertex tree with maximum number of vertices of degree one. The  $n$ -vertex path, denoted by  $\mathcal{P}_n$  is the  $n$ -vertex tree with minimum number of vertices of degree one. Finally, we denote the complete bipartite graph with two parts of sizes  $n_1$  and  $n_2$ , by  $\mathcal{K}_{n_1, n_2}$  (see [12] for more details and definitions).

## III. $\mathcal{H}_2$ NORM AS A ROBUSTNESS MEASURE

Consider a linear time-invariant system driven by unit variance white stochastic process

$$\begin{aligned} \dot{x} &= Ax + Bw, \\ y &= Cx, \quad x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^m, \end{aligned} \quad (2)$$

where  $w$  is an  $n$ -vector of zero-mean white noise process. In the case  $A$  is not necessarily Hurwitz, the state  $x$  may not have finite steady state variance. However, when the unstable modes of  $A$  do not observe from  $y$ , the output  $y$  has a finite steady state variance [1].

$$H := \lim_{t \rightarrow \infty} \mathbf{E}[y^T(t)y(t)] = \lim_{t \rightarrow \infty} \mathbf{E}[\|x(t)\|_{CTC}^2]. \quad (3)$$

For the state-space system (2), the  $\mathcal{H}_2$  norm is  $[\text{Tr}(CPC^T)]^{\frac{1}{2}}$ , where  $P$  is the solution of

$$AP + PA^T + BB^T = 0. \quad (4)$$

Also for calculating the  $\mathcal{H}_2$  norm, we can use the observability Grammian  $Q$ ,

$$QA + A^T Q + C^T C = 0, \quad (5)$$

and the  $\mathcal{H}_2$  norm is  $[\text{Tr}(B^T Q B)]^{\frac{1}{2}}$ . It is well known that the  $\mathcal{H}_2$  norm of system (2) is the same as the the steady state standard deviation of the state  $x$ , as given in [1], [2].

## IV. FUNDAMENTAL LIMITS ON ROBUSTNESS MEASURES OF NETWORKS WITH LINEAR DYNAMICS

In this section, we consider robustness of networks with linear dynamics with respect to external stochastic disturbance. We now seek to describe the robustness of these networks. For this aim we measure the robustness of the system by explicitly calculating the steady-state variance of the state.

We now state the first theorem of this paper. The following theorem present our main result on the bounds of the proposed performance measure.

*Theorem 1 (Main Result):* Consider the following linear dynamics

$$\dot{x} = Ax + w, \quad (6)$$

where  $A$  is Hurwitz and  $w(t) \in \mathbb{R}^n$  is a unit variance white stochastic process. Then we have

$$\begin{aligned} -\sum_{i=1}^n \frac{1}{2\text{Re}\{\lambda_i(A)\}} &\leq H = \lim_{t \rightarrow \infty} \mathbf{E}[x(t)^T x(t)] \\ &\leq -\sum_{i=1}^n \frac{1}{\lambda_i(A_s)}, \end{aligned} \quad (7)$$

where  $A_s = A^T + A$ .

*Proof:* Due to space limitations, we eliminate the proof. ■

In the following theorem we present that how the robustness index depends on the properties of  $A$  and the size of the network.

*Theorem 2:* Consider the linear dynamics (6) then we have

$$H \geq \sum_{i=1}^n -\frac{1}{2\text{Re}\{\lambda_i(A)\}} \geq \frac{n^2}{\sqrt{n \left( \frac{\|A\|_F^2 + \text{Tr}(A^2)}{2} \right)}}. \quad (8)$$

*Proof:* Due to space limitations, we eliminate the proof. ■

## V. CYCLIC INTERCONNECTION STRUCTURES

In this section, we show the utility of Theorem 1 for a class of systems with a cyclic interconnection structure. One typical example of such networks is a sequence of biochemical reactions where the system's product (output) is necessary to power and catalyze the first reaction.

Consider a LTI system  $\mathcal{L}_i$  represented by a state-space model of the form

$$\dot{x}_i = -a_i x_i + u_i, \quad y_i = c_i x_i,$$

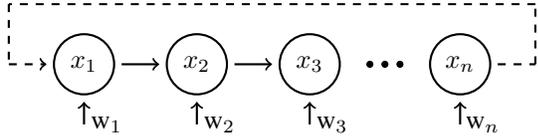


Fig. 1: Schematic diagram of negative feedback noisy cyclic system. The dashed link indicates a negative (inhibitory) feedback signal.

$$a_i, c_i > 0, \text{ for } 1 \leq i \leq n \quad (9)$$

where  $u_i(t)$ ,  $y_i(t)$  and  $x_i(t)$  denote its input, output and state respectively. We now consider a cyclic interconnection of dynamical systems  $\mathcal{L}_i, i = 1, 2, \dots, n$  with input  $u_i \in \mathbb{R}$  and output  $y_i \in \mathbb{R}$  (as depicted in Fig. 1),

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 - y_n + w_1, \\ \dot{x}_2 &= -a_2 x_2 + y_1 + w_2, \\ &\dots \\ \dot{x}_n &= -a_n x_n + y_{n-1} + w_n, \end{aligned} \quad (10)$$

where  $w_i$ 's are independent white random processes of the same intensity. The resulting dynamics are

$$\dot{x}(t) = Ax(t) + w(t), \quad (11)$$

where

$$A = \begin{bmatrix} -a_1 & 0 & \dots & 0 & -c_n \\ c_1 & -a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -a_{n-1} & 0 \\ 0 & 0 & \dots & c_{n-1} & -a_n \end{bmatrix}, \quad (12)$$

and  $w(t) \in \mathbb{R}^n$  is a unit variance white stochastic process. We now seek to describe the robustness of the cyclic interconnected network (11) subject to stochastic disturbances. For this aim we measure the robustness of the system by explicitly calculating the steady state variance of the state.

*Theorem 3:* Consider the cyclic interconnected network driven by a white random process (11), where  $\frac{c_1 c_2 \dots c_n}{a_1 a_2 \dots a_n} < \sec^n(\frac{\pi}{n})$  then for the steady state variance of the state we have

$$-\sum_{i=1}^n \frac{1}{2\text{Re}\{\lambda_i(A)\}} \leq H \leq -\sum_{i=1}^n \frac{1}{\lambda_i(A_s)}, \quad (13)$$

where  $A_s = A^T + A$ . In addition, when  $a := a_1 = \dots = a_n$  then

$$H \geq -\sum_{i=1}^n \frac{1}{2\text{Re}\{\lambda_i(A)\}} = \begin{cases} \frac{n \tan \frac{\beta}{2}}{2r \sin \frac{\beta}{2}}, & q < 1 \\ \frac{n}{2r}, & q = 1 \\ \frac{n \tanh \frac{\beta}{2}}{2r \sinh \frac{\beta}{2}}, & q > 1 \end{cases} \quad (14)$$

where  $r = \sqrt{c_1 c_2 \dots c_n}$ ,  $q = \frac{a}{r}$  and

$$\beta := \begin{cases} \arccos(q)n, & q \leq 1 \\ \text{arcosh}(q)n, & q > 1 \end{cases} \quad (15)$$

*Proof:* See [4]. ■

*Remark 1:* The classical secant criterion [13], [14] for (11) when  $a_1 = a_2 = \dots = a_n$ , implies that, the unperturbed

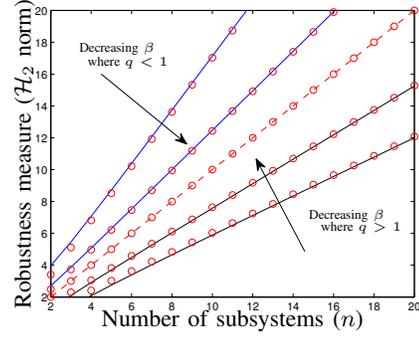


Fig. 2: The small circles ( $\circ$ ) demonstrate the lower bound of  $\mathcal{H}_2$  norm of interconnected networks (11), versus the number of subsystems. The solid lines show the obtained hard limits based on the approximation (16).

system (11) is stable if and only if  $q > \cos(\frac{\pi}{n})$ . So, in the case where  $\beta$  (15) is fixed, changing the number of intermediate subsystems does not change the stability behavior of the cyclic network. But, according to Theorem 3, in this case, the lower bound of  $H$  increases when the size of network increases. It is straightforward to show that the lower bound of  $H$  is in  $O(n^2)$ , where  $\beta$  is fixed, and it is approximated by

$$H \geq \begin{cases} \frac{\tan \frac{\beta}{2}}{2r\beta} n^2, & q < 1 \\ \frac{1}{2r} n^2, & q = 1 \\ \frac{\tanh \frac{\beta}{2}}{2r\beta} n^2, & q > 1 \end{cases} \quad (16)$$

This means that as the number of intermediate reactions  $n$  grows, the price paid for robustness,  $\mathcal{H}_2$  norm, increases linearly with  $n$  (see Fig. 2).

## VI. ROBUSTNESS OF NETWORKS WITH FIRST-ORDER CONSENSUS DYNAMICS

In this section we consider a first-order consensus dynamics over a weighted connected network modeled by  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$  with  $|\mathcal{V}(\mathcal{G})| = n$  and  $|\mathcal{E}(\mathcal{G})| = m$ . In the first-order setting, each node has a single state. So, the state of the system is a vector  $x \in \mathbb{R}^n$ . Our goal is to quantify the robustness of these networks to disturbances using the quantity that we call  $L_Q$ -energy. Finally, the relation of this quantity and the graph structure is studied in this section. Consider the linear consensus dynamics with

$$\dot{x} = -Lx + w, \quad (17)$$

where  $L$  is the Laplacian matrix of the underlying graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ ,  $x$  is a corresponding state vector of the agents and each node state subject to stochastic disturbance.

*Definition 1:* The  $L_Q$ -energy of the first-order network (17) is defined as the expected steady state  $L_Q$ -norm of state  $x$

$$H_{L_Q}^{(1)} = \lim_{t \rightarrow \infty} \mathbf{E} [\|x(t)\|_{L_Q}^2]$$

$$= \lim_{t \rightarrow \infty} \mathbf{E} \left[ \sum_{(u,v) \in \mathcal{E}_{\mathcal{Q}}} \omega_{u,v}^{(\mathcal{Q})} (x_u(t) - x_v(t))^2 \right], \quad (18)$$

where  $L_{\mathcal{Q}}$  is the Laplacian matrix of weighted graph  $\mathcal{Q} = (\mathcal{V}, \mathcal{E}_{\mathcal{Q}}, \mathcal{W}_{\mathcal{Q}})$  and  $\omega_{u,v}^{(\mathcal{Q})}$  is the weight of edge  $(u, v)$  in the graph  $\mathcal{Q}$ . So, the  $L_{\mathcal{Q}}$ -energy of (17), is the expected steady state Laplacian quadratic form of  $\mathcal{Q}$ .

*Theorem 4:* Consider the linear consensus dynamics (17), then we have  $H_{L_{\mathcal{Q}}}^{(1)} = \frac{1}{2} \text{Tr}(L_{\mathcal{Q}} L^+)$ .

*Proof:* Due to space limitations, we eliminate the proof. ■

*Remark 2:* We note that when  $\mathcal{K}_n$  is a complete graph with each edge weight equal to  $\frac{1}{n}$ , then  $L_{\mathcal{K}_n} = I - \frac{1}{n}J$ . And the  $L_{\mathcal{K}_n}$ -energy of the network (17) is

$$H^{(1)} = H_{L_{\mathcal{K}_n}}^{(1)} = \sum_{i=2}^n \frac{1}{2\lambda_i}. \quad (19)$$

And the  $L_{\mathcal{K}_n}$ -energy of the first-order network reduces to the concept of first-order network coherence and the expected dispersion of the system in steady state [1], [2]. It has been shown that the total effective resistance depends on the spectrum of the Laplacian matrix as  $R = 2n \sum_{i=2}^n \frac{1}{\lambda_i}$  (see [1] for more details). Hence based on this fact and (14) we have

$$H^{(1)} = \frac{R}{4n}. \quad (20)$$

Since  $H^{(1)}$  can be computed using only the communication graph  $\mathcal{G}$  (the Laplacian matrix  $L$ ), in the rest of this paper we associate the  $H^{(1)}$  with the graph  $\mathcal{G}$ .

## VII. GRAPH-DEPENDENT FUNDAMENTAL LIMITS ON ROBUSTNESS MEASURE $H^{(1)}$

In this section we assume a simple case of first-order consensus dynamics over a simple connected network modeled by  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \{1\})$ . Also we just consider  $L_{\mathcal{K}_n}$ -energy, which is coincide with the steady state variance of state  $x$ , i.e., expected dispersion of the system in steady-state. In the following we present some graph-dependent fundamental limits on  $H^{(1)}$ .

### A. Lower and Upper Bounds

First, we present some results on the lower and upper bounds of the  $L_{\mathcal{K}_n}$ -energy (i.e.,  $H^{(1)}$ ). Note that one of the important graph parameters is the diameter of the graph, which is defined by the largest distance between any two graph vertices.

*Theorem 5:* Consider the linear consensus dynamics (17), then we have

$$\begin{aligned} \frac{-n^2}{2\text{Tr}(L)} &\leq \frac{n^{1.5}}{2\|L\|_F} = \frac{n^{1.5}}{2\sqrt{s_1 + s_2}} \leq H^{(1)} \\ &\leq \frac{1}{2} \left[ 1 + \left( \binom{n}{2} - m \right) \text{diam}(\mathcal{G}) \right], \end{aligned} \quad (21)$$

where  $\|\cdot\|_F$  denotes Frobenius norm,  $s_r := \sum_{i=1}^n d(i)^r$  and  $d(i)$  is the degree of node  $i$ .

*Proof:* The first inequality in (21) is obtained from Theorem 2 and the upper bound inequality in (21) can be obtained by using (20) and Theorem 5.6 in [15]. ■

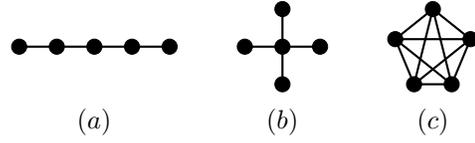


Fig. 3: (a)  $\mathcal{P}_5$  has maximal  $H^{(1)}$  index among all connected graph of order 5 (b)  $\mathcal{S}_5$  has minimal  $H^{(1)}$  index among all trees of order 5 (c)  $\mathcal{K}_5$  has minimal  $H^{(1)}$  index among all graph of order 5.

The following theorem indicates the maximal and minimal robustness index graphs of order  $n$  (see Fig. 3).

*Theorem 6:* Consider the linear consensus dynamics (17), then we have

$$\frac{(n-1)}{2n} \leq H^{(1)} \leq \frac{n^2-1}{12}. \quad (22)$$

With lower equality if and only if  $\mathcal{G}$  is a complete graph and upper equality if and only if  $\mathcal{G}$  is a path.

*Proof:* Due to space limitations, we eliminate the proof. ■

*Theorem 7:* Let  $\mathcal{G}$  be a connected graph with  $n$  vertices and  $\mathcal{P}$  a connected spanning subgraph of  $\mathcal{G}$ . Then  $H^{(1)}(\mathcal{G}) \leq H^{(1)}(\mathcal{P})$ , with equality if and only if  $\mathcal{G} = \mathcal{P}$ .

*Proof:* Due to space limitations, we eliminate the proof. ■

### B. The Characteristic Polynomial

We now show the relation between the coefficient of the following characteristic polynomial of the Laplacian matrix  $L$  and  $L_{\mathcal{K}_n}$ -energy,

$$\Phi(L, \lambda) = \sum_{k=0}^n (-1)^{n-k} c_k(L) \lambda^k. \quad (23)$$

The connection between the coefficients of the Laplacian characteristic polynomial and the structure of the graph was established by Kelmans [16]

$$c_k(L) = \sum_{F \in \mathcal{F}_k(\mathcal{G})} \gamma(F), \quad (24)$$

where  $F$  is a spanning forest,  $\mathcal{F}_k(\mathcal{G})$  is the set of all spanning forests of  $\mathcal{G}$  with exactly  $k$  components, and  $\gamma(F)$  is the product of the number of vertices of the components of  $F$  [17].

From (19) and Vietes formula for (23) we get

$$H^{(1)} = \frac{c_2(L)}{2c_1(L)}. \quad (25)$$

In addition, based on (24) the number of spanning trees of  $\mathcal{G}$  is

$$\mathbf{T}(\mathcal{G}) = \frac{1}{n} \lambda_2 \cdots \lambda_n = \frac{1}{n} c_1(L). \quad (26)$$

Hence, from (25) and (26) it follows that  $H^{(1)} = \frac{c_2(L)}{2n\mathbf{T}(\mathcal{G})}$ .

One method for computation of the coefficient of (23) is the recursive algorithm. The following recursive formulae is devised by Fadeev

$$c_{n-k} = \frac{1}{k} \text{Tr}(L_k), \quad (27)$$

where  $L_k = L^k - (-1)^{k-1}c_{n-1}L^{k-1} - \dots - (-1)^1c_{n-k+1}L$  (see [18] for more details). The following theorem shows the relation between the number of spanning trees and  $H^{(1)}$ .

**Theorem 8:** Consider the linear consensus dynamics (17), then we have  $\frac{n-1}{2^{n-1}\sqrt{n\mathbf{T}(\mathcal{G})}} \leq H^{(1)} = \frac{c_2(L)}{2n\mathbf{T}(\mathcal{G})}$ .

*Proof:* Due to space limitations, we eliminate the proof. ■

### VIII. ROBUSTNESS OF NETWORKS WITH SECOND-ORDER DYNAMICS

In this section we consider a second-order consensus dynamics. Let us consider the following network with second-order dynamics.

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ F & G \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w, \quad (28)$$

where  $G$  and  $F$  are the linear feedback operators and  $w$  is a  $2n$ -vector of zero-mean white noise processes [1]. To quantify the robustness of these networks to disturbances we define the following measures.

**Definition 2:** The position  $L_{\mathcal{Q}}$ -energy of the second-order network (28) is defined as the expected steady state  $L_{\mathcal{Q}}$ -norm of position vector  $x$

$$\begin{aligned} H_{x,\mathcal{Q}}^{(2)} &= \lim_{t \rightarrow \infty} \mathbf{E} [\|x(t)\|_{L_{\mathcal{Q}}}^2] \\ &= \lim_{t \rightarrow \infty} \mathbf{E} \left[ \sum_{(i,j) \in \mathcal{E}_{\mathcal{Q}}} \omega_{i,j}^{(\mathcal{Q})} (x_i(t) - x_j(t))^2 \right], \quad (29) \end{aligned}$$

where  $L_{\mathcal{Q}}$  is the Laplacian matrix of weighted graph  $\mathcal{Q} = (\mathcal{V}, \mathcal{E}_{\mathcal{Q}}, \mathcal{W}_{\mathcal{Q}})$ .

**Definition 3:** The velocity  $L_{\mathcal{Q}}$ -energy of the second-order network (28) is defined as the expected steady state  $L_{\mathcal{Q}}$ -norm of velocity vector  $v$

$$\begin{aligned} H_{v,\mathcal{Q}}^{(2)} &= \lim_{t \rightarrow \infty} \mathbf{E} [\|v(t)\|_{L_{\mathcal{Q}}}^2] \\ &= \lim_{t \rightarrow \infty} \mathbf{E} \left[ \sum_{(i,j) \in \mathcal{E}_{\mathcal{Q}}} \omega_{i,j}^{(\mathcal{Q})} (v_i(t) - v_j(t))^2 \right], \quad (30) \end{aligned}$$

where  $L_{\mathcal{Q}}$  is the Laplacian matrix of weighted graph  $\mathcal{Q} = (\mathcal{V}, \mathcal{E}_{\mathcal{Q}}, \mathcal{W}_{\mathcal{Q}})$ .

The operators  $G$  and  $F$  will have some very special structure depending on assumptions of the type of feedback and measurements available [1]. Here we consider two cases.

**Case 1:** In this case we assume that,  $G = -\beta I$  and  $F = -L$ , where  $L$  is the Laplacian matrix of the connected graph  $\mathcal{G}$ . Hence, we have

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L & -\beta I \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w. \quad (31)$$

The position  $L_{\mathcal{Q}}$ -energy of (31) is obtained by

$$H_{x,L_{\mathcal{Q}}}^{(2)} = \frac{1}{2\beta} \mathbf{Tr}(L_{\mathcal{Q}}L^+). \quad (32)$$

In addition, the velocity  $L_{\mathcal{Q}}$ -energy of (31) is obtained by

$$H_{v,L_{\mathcal{Q}}}^{(2)} = \frac{1}{2\beta} \mathbf{Tr}(L_{\mathcal{Q}}). \quad (33)$$

**Case 2:** In this case we assume that,  $G = -\beta L$  and  $F = -L$  where  $L$  is the Laplacian of the connected graph  $\mathcal{G}$ . Hence, we get

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L & -\beta L \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w. \quad (34)$$

The position  $L_{\mathcal{Q}}$ -energy of (34) is

$$H_{x,L_{\mathcal{Q}}}^{(2)} = \frac{1}{2\beta} \mathbf{Tr}(L_{\mathcal{Q}}(L^+)^2). \quad (35)$$

In addition, the velocity  $L_{\mathcal{Q}}$ -energy of (34) is

$$H_{v,L_{\mathcal{Q}}}^{(2)} = \frac{1}{2\beta} \mathbf{Tr}(L_{\mathcal{Q}}L^+). \quad (36)$$

### IX. APPLICATIONS OF OUR RESULTS

Here we present two general examples to demonstrate the utility of the proposed performance measure in Section VIII.

#### A. Power Networks

Consider a power network with underlying graph  $\mathcal{G}$  which consists of  $n$  buses (nodes) and  $m$  transmission lines (edges). At each node with index  $i$ , there is a generator  $G_i$  with inertia constant  $M_i$ , damping constant  $\beta_i$ , voltage magnitude  $V_i$ , and angle  $\theta_i$ . Let us denote the admittance over edge  $e \in \mathcal{E}(\mathcal{G})$  by  $y_e = g_e - \mathbf{j}b_e$  where  $g_e$  and  $b_e$  are, respectively, the conductance and susceptance of the transmission line corresponding to edge  $e \in \mathcal{E}(\mathcal{G})$ . The conductance graph  $\mathcal{G}_g$  and the susceptance graph  $\mathcal{G}_b$  are copies of graph  $\mathcal{G}$ , but with new weights  $g_e$  and  $b_e$  on their edges, respectively. Let us define  $\alpha_e$  as the ratio of the conductance to susceptance of edge  $e$ , i.e.,  $\alpha_e = \frac{g_e}{b_e}$ . We can linearize the swing equations around the zero operating point and obtain

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L_B & -\beta I \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w. \quad (37)$$

We refer to [9] for more details. The resistive power loss over edge connecting nodes  $i$  and  $j$  is given by

$$P_{\text{loss}}^{i,j} = g_{i,j} |V_i - V_j|^2. \quad (38)$$

Therefore, the total resistive power loss of the network is

$$P_{\text{loss}} = \sum_{(i,j) \in \mathcal{E}(\mathcal{G})} g_{i,j} |V_i - V_j|^2. \quad (39)$$

Using a small angle approximation, one gets

$$\tilde{P}_{\text{loss}} = \sum_{(i,j) \in \mathcal{E}(\mathcal{G})} g_{i,j} |\theta_i - \theta_j|^2. \quad (40)$$

According to Definition 2 and (40), it follows that the position (in fact, angle)  $L_{\mathcal{G}_g}$ -energy of (37) and the resistive power loss  $\tilde{P}_{\text{loss}}$  are equivalent.

**Theorem 9:** The  $L_{\mathcal{G}_g}$ -energy of the linearized swing dynamics (37), i.e., the resistive power loss, is lower and upper bounded by

$$\frac{\alpha_{\min}}{2\beta} (n-1) \leq H_{\theta,L_{\mathcal{G}_g}}^{(2)} = \frac{\bar{\alpha}}{2\beta} (n-1) \leq \frac{\alpha_{\max}}{2\beta} (n-1), \quad (41)$$

where  $\alpha_{\min} = \min_{e \in \mathcal{E}(\mathcal{G})} \alpha_e$ ,  $\alpha_{\max} = \max_{e \in \mathcal{E}(\mathcal{G})} \alpha_e$  and  $\bar{\alpha}$  is a weighted mean of  $\alpha_e$  over all edges.

*Proof:* Due to space limitations, we eliminate the proof. ■

*Remark 3:* We note that when  $\alpha_1 = \dots = \alpha_m$ , the result of Theorem 9 reduces to the results given in [9].

*Remark 4:* Note from (41) that the total resistive losses scale with the product of the network size and the weighted mean of  $\alpha_i$ 's (i.e., the ratios of line resistances to their reactances). The weighted mean depends on the networks topology, but the lower and upper bounds of the total resistive losses are independent of the network's topology, and they scale unboundedly with the number of generators.

*Theorem 10:* For edge-transitive networks with the internal conductance common to all edges we have

$$H_{\theta, L_{\mathcal{G}_g}}^{(2)} = \frac{\sum_{e \in \mathcal{E}(\mathcal{G})} \alpha_e}{2\beta m} (n-1). \quad (42)$$

*Proof:* Due to space limitations, we eliminate the proof. ■

An edge-transitive graph is a graph  $\mathcal{G}$  such that, given any two edges  $e_1$  and  $e_2$  of  $\mathcal{G}$ , there is an automorphism of  $\mathcal{G}$  that maps  $e_1$  to  $e_2$ . Hence, a graph is edge-transitive if every edge has the same local environment. Note that all biregular, star, cycle and complete graphs are edge-transitive [19].

*Theorem 11:* For tree networks we have

$$H_{\theta, L_{\mathcal{G}_g}}^{(2)} = \frac{\sum_{e \in \mathcal{E}(\mathcal{G})} \alpha_e}{2\beta m} (n-1). \quad (43)$$

*Proof:* Due to space limitations, we eliminate the proof. ■

## B. Vehicle Formation Problems

In the vehicle formation problem, there are  $n$  vehicles, each with a position and a velocity. The objective is for each vehicle to travel at a constant target velocity while maintaining a fixed, pre-specified distance between itself and each of its neighbors [8].

Here we can assume that the system dynamics are given by (34) and  $\beta = 1$ . The quantity  $H_{v,L}^{(2)}$  coincides with the energy of the flock which is defined by  $\sum_{i,j} w_{i,j} (v_i - v_j)^2$  in [10]. Now according to (36) we have

$$H_{v,L}^{(2)} = \frac{1}{\beta} \text{Tr}(L^+) = \sum_{i=2}^n \frac{1}{2\lambda_i}. \quad (44)$$

We now consider the velocity  $L_{\mathcal{K}_n}$ -energy of (34) where  $\mathcal{K}_n$  is a complete graph with each edge weight equal to  $\frac{1}{n}$ . Then we have

$$H^{(2)} := H_{x, L_{\mathcal{K}_n}}^{(2)} = \sum_{i=2}^n \frac{1}{2\lambda_i^2}. \quad (45)$$

In this case  $H^{(2)}$  coincides with the second-order network coherence which is defined in [8]. Since  $H^{(2)}$  can be computed using only the graph  $\mathcal{G}$  (the Laplacian matrix  $L$ ), in the rest of this paper we associate the  $H^{(2)}$  with the graph  $\mathcal{G}$ . Hence here we just consider the relation between  $H^{(2)}$  and the underlying graph structure.

*Theorem 12:* Let  $\mathcal{G}$  be a connected graph with  $n$  vertices and  $\mathcal{P}$  a connected spanning subgraph of  $\mathcal{G}$ . Then,  $H^{(2)}(\mathcal{G}) \leq H^{(2)}(\mathcal{P})$  with equality if and only if  $\mathcal{G} = \mathcal{P}$ .

*Proof:* Due to space limitations, we eliminate the proof. ■

*Theorem 13:* Consider the second-order linear consensus dynamics (28), then we have  $\frac{(2m)^4}{2(s_2+s_1)^3} \leq H^{(2)}$ .

*Proof:* Due to space limitations, we eliminate the proof. ■

## X. CONCLUSION

In this paper, we exploit structural properties of networks of interconnection systems in order to characterize their robustness properties and fundamental limits. Our first focus is to measure robustness of networks with first-order dynamics in presence of external stochastic disturbances. We explicitly calculate the  $L_Q$ -energy of the network, which measures the expected steady state  $L_Q$ -norm of the state of the entire network. We show that the introduced robustness measure depends on characteristics of the underlying graph of the network and the graph  $Q$ . Finally, we generalize our results for networks with second-order dynamics.

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