Sparsity Measures for Spatially Decaying Systems*

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***Abstract*— We consider the omnipresent class of spatially decaying systems, where the sensing and controls is spatially distributed. This class of systems arises in various applications where there is a notion of spatial distance with respect to which couplings between the subsystems can be quantified using a class of coupling weight functions. We exploit spatial decay property of the dynamics of the underlying system in order to introduce system-oriented sparsity measures for spatially distributed systems. We develop a new mathematical framework, based on notions of quasi-Banach algebras of spatially decaying matrices, to relate spatial decay properties of spatially decaying systems to sparsity features of their underlying information structures. Moreover, we show that the inverse-closedness property of matrix algebras plays a central role in exploiting various structural properties of spatially decaying systems. We show that the quadratically optimal state feedback controllers for spatially decaying systems are sparse and spatially localized in the sense that they have near-optimal sparse information structures. Finally, our results are applied to quantify sparsity and spatial localization features of a class of randomly generated power networks.**

I. Introduction

The emergence of a new theory of compressive feedback control design for spatially distributed systems within the areas of control systems, operational research, and machine learning is eminent. There are a number of substantial challenges that need to be overcome, in order to reap the full potential of this rapidly growing emergent area. The primary challenge is to define meaningful sparsity measures which are specifically tailored for spatially distributed dynamical systems and their applications. A viable sparsity measure should capture both *sparsity* and *spatial localization* features of spatially distributed dynamical networks. Therefore, our primary goal in this paper is to explore some of the fundamental insights and tools that will allow us to exploit architectural properties of the underlying systems in order to introduce system-oriented sparsity measures for spatially distributed systems.

Our main focus will be on an important class of spatially distributed systems, so called *spatially decaying systems*, for which the corresponding optimal controllers have an inherently semi-decentralized information structures [1]–[4]. This class of systems have important spatial and temporal aspects. We refer the reader to the seminal work of Bamieh *et al.* for a thorough literature review of spatio-temporal systems [1]. The methods and concepts of spatio-temporal systems are greatly generalized in [1] where it is shown that for systems with similar spatial symmetries, it is possible to achieve optimal performance without losing symmetry. More recent related works with a similar mindset were focused on localization of optimal controllers for spatially invariant systems [5]–[7]. A recent results on employing iterative augmented Lagrangian approach and alternating direction method of multipliers to design sparse optimal controller is reported in [8] and [9].

In this paper, we introduce the family of Gröchenig-Schur class of spatially decaying matrices equipped with a quasi–norm, which is so called $\mathcal{S}_{q,w}$–measure for a given coupling weight function w. The class of $\mathcal{S}_{q,w}$–measures are defined using ℓ^q–norms for $0 < q \leq \infty$ and are indeed asymptotic approximations of the ideal sparsity measure as q tends to zero (see Sections II and III). This novel framework greatly generalizes our earlier works [3] and [2] which only consider the class of linear systems with bounded $\mathcal{S}_{1,w}$–norm (i.e., $q = 1$). We show that distributed optimal control problems that defined over the Gröchenig-Schur class of matrices have an inherently semi-decentralized information structures [4]. This architecture determines the communication requirements in the controller array. We show that $\mathcal{S}_{q,w}$–measure is an appropriate measure to quantify sparsity and spatial locality features of information structures of this class of systems (see Section VI).

We take the first step towards development of a rigorous mathematical foundation to study sparsity in spatially decaying systems by proposing the abstract notion of q-Banach algebras for $0 < q \leq 1$ (see Section IV). A q-Banach algebra with exponent $0 < q < 1$ is not a Banach space, which prevents us from employing existing techniques to analyze spatially distributed systems. All existing proof methods to study algebraic properties of solutions of algebraic Lyapunov and Riccati equations cannot be directly applied for systems that are defined over a q-Banach algebra for $0 < q < 1$. The range of exponents $0 < q < 1$ is extremely important for us as we can asymptotically approximate sparsity and spatial localization features of spatially decaying systems as q tends to zero (see Section VI). We characterize the class of *proper* q-Banach algebras and show that the unique solutions of the algebraic Lyapunov and Riccati equations which are defined over a proper q-Banach algebra also belong to that q-Banach algebra, significantly generalizing all existing works in the literature (e.g., see [1]–[4], [10], [12], [13], [30]–[32] and reference in there) to consider spatially distributed systems that are defined over q-Banach algebras for $0 < q \leq \infty$.

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Mathematical Notations. Throughout the paper, the underlying discrete spatial domain of a spatially distributed system is denoted by $\mathbb{G}$ which is a subset of $\mathbb{Z}^d$ for $d \geq 1$. For a given discrete spatial domain $\mathbb{G}$, the ℓ^0–measure of a vector $x = (x_i)_{i\in\mathbb{G}}$ is defined as $\|x\|_{\ell^0(\mathbb{G})} := \mathbf{card}\big\{x_i \neq 0 \bigm| i \in \mathbb{G}\big\}$, where **card** is the cardinality of a set. The ℓ^0–measure is an ideal sparsity measure for various applications whose value represents the total number of nonzero entries in a vector. The ℓ^q–measure of x is defined by $\|x\|^q_{\ell^q(\mathbb{G})} = \sum_{i\in\mathbb{G}} |x_i|^q$ for all $0 < q < \infty$, and $\|x\|_{\ell^\infty(\mathbb{G})} = \sup_{i\in\mathbb{G}} |x_i|$ for $q = \infty$.

II. Generalized Spatially Decaying Systems

In order to model couplings between subsystems in a spatially distributed systems, we introduce a class of coupling weight functions. This class of weight functions are more general than the class of weigh functions introduced in [17] which requires verification of a rather restrictive condition so called Gelfand-Raikov-Shilov (GRS) condition.

A. Admissible Coupling Weight Functions

Definition 1: Let ρ be a real-valued function on $\mathbb{G} \times \mathbb{G}$. Then, ρ is a quasi-distance function on $\mathbb{G}$ if it satisfies:

(i) $\rho(i, j) \geq 0$ for all $i, j \in \mathbb{G}$;
(ii) $\rho(i, j) = 0$ if and only if $i = j$; and
(iii) $\rho(i, j) = \rho(j, i)$ for all $i, j \in \mathbb{G}$.

A quasi-distance function is different from a distance function in that the triangle inequality is not required to hold. In case, $\mathbb{G}$ corresponds to an unweighted undirected graph, one may use the shortest distance on a graph to define the quasi-distance $\rho(i, j)$ from vertex i to vertex j.

Definition 2: A positive function w on $\mathbb{G}\times\mathbb{G}$ is a weight function if it satisfies:

(i) $w(i, j) \geq 1$ for all $i, j \in \mathbb{G}$;
(ii) $w(i, j) = w(j, i)$ for all $i, j \in \mathbb{G}$; and
(iii) $\sup_{i\in\mathbb{G}} w(i, i) < \infty$.

Definition 3: A weight function w on $\mathbb{G} \times \mathbb{G}$ is submultiplicative if there exists a positive constant K_0 such that

$$w(i, j) \;\leq\; K_0\, w(i, k)\, w(k, j) \tag{1}$$

for all $i, j, k \in \mathbb{G}$.

Definition 4: For $0 < q \leq 1$, suppose that w is a coupling weight function on $\mathbb{G} \times \mathbb{G}$ and $\rho : \mathbb{G} \times \mathbb{G} \longmapsto \mathbb{R}$ is a quasi-distance function. The coupling weight function w is called admissible whenever there exist a companion weight function $u = (u(i, j))_{i,j\in\mathbb{G}}$, an exponent $\theta \in (0, 1)$, and a positive constant D such that

$$w(i, j) \;\leq\; w(i, k)\, u(k, j) \;+\; u(i, k)\, w(k, j) \tag{2}$$

for all $i, j, k \in \mathbb{G}$, and

$$\sup_{i\in\mathbb{G}} \inf_{\tau\geq 0} \Big(\sum_{\substack{j\in\mathbb{G}\\ \rho(i,j)<\tau}} |u(i, j)|^{\frac{1}{1-\frac{q}{2}}} \Big)^{1-\frac{q}{2}} + t \sup_{\substack{j\in\mathbb{G}\\ \rho(i,j)\geq\tau}} \frac{u(i, j)}{w(i, j)} \leq D t^{1-\theta}$$

for all $t \geq 1$.

Example 5: The two omnipresent admissible coupling weight functions which appear in modeling of most real-world systems are sub-exponential coupling weight functions

$$e_{\sigma,\delta} := \big(e_{\sigma,\delta}(i, j)\big)_{i,j\in\mathbb{Z}^d} = \left(\exp\left(\left(\frac{\|i - j\|_\infty}{\sigma}\right)^\delta\right)\right)_{i,j\in\mathbb{Z}^d}$$

with parameters $\sigma > 0$ and $\delta \in (0, 1)$, and the class of polynomial coupling weight functions

$$\pi_{\alpha,\sigma} \;:=\; \big(\pi_{\alpha,\sigma}(i, j)\big)_{i,j\in\mathbb{Z}^d} \;=\; \left(\left(\frac{1 + \|i - j\|_\infty}{\sigma}\right)^\alpha\right)_{i,j\in\mathbb{Z}^d}$$

with parameters $\alpha, \sigma > 0$.

B. Generalized Class of Spatially Decaying Matrices

In this subsection, we introduce the most general class of spatially decaying matrices with respect to ℓ^q–measure for all $0 < q \leq \infty$.

Definition 6: For a given admissible weight function w on $\mathbb{G} \times \mathbb{G}$, the family of Gröchenig-Schur class of matrices on $\mathbb{G}$ is denoted by $\mathcal{S}_{q,w}(\mathbb{G})$ and defined as

$$\mathcal{S}_{q,w}(\mathbb{G}) = \Big\{A \;\Big|\; \|A\|_{\mathcal{S}_{q,w}(\mathbb{G})} < \infty\Big\} \tag{3}$$

for $0 < q \leq \infty$. For $0 < q < \infty$, the matrix norm (in fact, it is quasi-norm for $0 < q < 1$) is defined by

$$\|A\|_{\mathcal{S}_{q,w}(\mathbb{G})} := \max\Big\{ \sup_{i\in\mathbb{G}} \Big(\sum_{j\in\mathbb{G}} |a(i, j)|^q w(i, j)^q \Big)^{1/q}, \sup_{j\in\mathbb{G}} \Big(\sum_{i\in\mathbb{G}} |a(i, j)|^q w(i, j)^q \Big)^{1/q} \Big\}, \tag{4}$$

and for $q = \infty$ by

$$\|A\|_{\mathcal{S}_{\infty,w}(\mathbb{G})} := \sup_{i,j\in\mathbb{G}} |a(i, j)| w(i, j).$$

We should emphasize that some subclasses of the family of Gröchenig-Schur class of spatially decaying matrices were previously introduced for $q = 1$ in [2], [3], [17] and for $1 \leq q \leq \infty$ in [20], [21]. When $0 < q < 1$, quantity (4) becomes a quasi-norm as ℓ^q-measures are nonconvex functions for this range of exponents. In [4], we verified the subalgebra properties of this class of matrices.

III. Ideal Sparsity Measure and Its Asymptotics

The ideal sparsity measure for vectors can be used to define a new sparsity measure for a matrix $A = (a(i, j))_{i,j\in\mathbb{G}}$ as follows

$$\|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} := \max\Big\{ \sup_{i\in\mathbb{G}} \|a(i, .)\|_{\ell^0(\mathbb{G})}, \sup_{j\in\mathbb{G}} \|a(., j)\|_{\ell^0(\mathbb{G})} \Big\},$$

where $a(i, .)$ is the i'th row and $a(., j)$ the j'th column of matrix A. The value of $\mathcal{S}_{0,1}$–measure reflects the maximum number of nonzero entries in all rows and columns of matrix A. The value of the $\mathcal{S}_{0,1}$–sparsity measure delivers some valuable information about sparsity as well as the spatial locality features of a given sparse matrix. The set of all sparse matrices with respect to a discrete spatial domain $\mathbb{G}$

is defined by

$$\mathcal{S}_{0,1}(\mathbb{G}) := \Big\{ A \;\Big|\; \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} < \infty \Big\}.$$

Proposition 7: The $\mathcal{S}_{0,1}$–measure satisfies the following properties:

(i) $\|\alpha A\|_{\mathcal{S}_{0,1}(\mathbb{G})} \leq \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})}$;
(ii) $\|A+B\|_{\mathcal{S}_{0,1}(\mathbb{G})} \leq \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} + \|B\|_{\mathcal{S}_{0,1}(\mathbb{G})}$; and
(iii) $\|AB\|_{\mathcal{S}_{0,1}(\mathbb{G})} \leq \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} \|B\|_{\mathcal{S}_{0,1}(\mathbb{G})}$,

for all scalars α and matrices $A, B \in \mathcal{S}_{0,1}(\mathbb{G})$.

Properties (i)-(iii) imply that the set $\mathcal{S}_{0,1}(\mathbb{G})$ is closed under addition and multiplication. However, it is not inverse-closed. For instance, a Toeplitz band matrix belongs to $\mathcal{S}_{0,1}(\mathbb{G})$ while its inverse may not live in $\mathcal{S}_{0,1}(\mathbb{G})$. In fact, $\mathcal{S}_{0,1}(\mathbb{G})$ is not even an algebra. We show that inverse-closedness property of matrix algebras plays a central role in exploiting various structural properties of spatially distributed systems that are defined over such matrix algebras.

A. Asymptotic Approximations

The following result motivates us to consider relaxations of the ideal $\mathcal{S}_{0,1}$–measure for spatially decaying matrices in order to quantitatively identify near–sparse information structures with semi-decentralized architectures. For every vector $x = (x_i)_{i \in \mathbb{G}}$ with bounded entries, ℓ^q–measure approximates the ℓ^0–measure asymptotically, i.e.,

$$\lim_{q \to 0} \; \|x\|^q_{\ell^q(\mathbb{G})} \;=\; \|x\|_{\ell^0(\mathbb{G})}. \tag{5}$$

This property enables us to approximate $\mathcal{S}_{0,1}$–measure by $\mathcal{S}_{q,w}$–measure asymptotically.

Theorem 8: For a given matrix $A = (a(i,j))_{i,j \in \mathbb{G}}$ with bounded $\mathcal{S}_{0,1}$–measure and bounded entries, i.e., $\|A\|_{\mathcal{S}_{\infty,w}} < \infty$, we have

$$\lim_{q \to 0} \; \|A\|^q_{\mathcal{S}_{q,w}(\mathbb{G})} \;=\; \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})}. \tag{6}$$

The most important implication of this theorem is that small values of q (closer to zero) can lead to reasonable approximations of the ideal sparsity measure for matrices. This is particularly true for spatially decaying matrices with slowly decaying rates, such as polynomially decaying matrices. However, for matrices with rapidly decaying rates, such as sub-exponentially decaying matrices, larger values of q (closer to one) can also result in reasonable measures for sparsity (see Section VI).

IV. Mathematical Foundation of Sparsity in Spatially Decaying Systems

The basic properties of the family of Gröchenig-Schur class of matrices $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q \leq 1$ motivates us to generalize our results to more abstract ground in order to capture essence of notions of sparsity in spatially decaying systems. Some of these inherent fundamental properties include the subalgebra properties in Proposition 3.3 in [4] and the differential norm property as it is reported in [33].

A. Generalized Sparsity Measures via q-Banach Algebras

Definition 9: For $0 < q \leq 1$, a complex vector space of matrices $\mathcal{A}$ is a q-Banach algebra equipped with q-norm $\|\cdot\|_{\mathcal{A}}$ if it contains a unit element I, i.e., $M := \|I\|_{\mathcal{A}} < \infty$, and the q-norm satisfies

(i) $\|A\|_{\mathcal{A}} \geq 0$, and $\|A\|_{\mathcal{A}} = 0$ if and only if $A = 0$,
(ii) $\|\alpha A\|_{\mathcal{A}} = |\alpha| \, \|A\|_{\mathcal{A}}$,
(iii) $\|A+B\|^q_{\mathcal{A}} \leq \|A\|^q_{\mathcal{A}} + \|B\|^q_{\mathcal{A}}$,
(iv) $\|AB\|_{\mathcal{A}} \leq K_0 \|A\|_{\mathcal{A}} \|B\|_{\mathcal{A}}$

for all $A, B \in \mathcal{A}$ and all $\alpha \in \mathbb{C}$, where $K_0 > 0$.

Definition 10: For a given exponent $0 < q \leq 1$ and Banach algebra $\mathcal{B}$, its q-Banach subalgebra $\mathcal{A}$ is a *differential* subalgebra of order $\theta \in (0,1]$ if there exist a constant $D > 0$ such that its q-norm satisfies the differential norm property

$$\|AB\|^q_{\mathcal{A}} \;\leq\; D \, \|A\|^q_{\mathcal{A}} \, \|B\|^q_{\mathcal{A}} \left(\left(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}} \right)^{q\theta} + \left(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}} \right)^{q\theta} \right)$$

for all $A, B \in \mathcal{A}$.

For a Banach subalgebra $\mathcal{A}$ (i.e., when $q = 1$), the above differential norm property with $\theta \in (0,1]$ has been widely used in operator theory and noncommutative geometry [11], [19], [24], and in solving (non)linear functional equations [27], [28]. The differential norm property plays a crucial role in establishing inverse-closedness property for q-Banach algebra, identifying exponential stability conditions, and exploiting algebraic properties of the unique solutions of Lyaponuv and algebraic Ricatti equations in q-Banach algebras. One of the most interesting examples of a q-Banach algebra is $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q \leq 1$. It turns out that the q-Banach algebra $\mathcal{S}_{q,w}(\mathbb{G})$ enjoys the above differential norm property (see [33] for a complete proof).

Definition 11: For $0 < q \leq 1$, we say that $\mathcal{A}$ is a *proper* q-Banach algebra of matrices on $\mathbb{G}$ equipped with q-norm $\|\cdot\|_{\mathcal{A}}$ if it is a q-Banach algebra with the following additional properties:

(P1) $\mathcal{A}$ is closed under the complex conjugate operation, i.e., for all $A \in \mathcal{A}$ we have

$$\|A^*\|_{\mathcal{A}} = \|A\|_{\mathcal{A}}; \tag{7}$$

(P2) $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$ and continuously embedded w.r.t. it, i.e., for all $A \in \mathcal{A}$ we have

$$\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; \|A\|_{\mathcal{A}}; \tag{8}$$

(P3) $\mathcal{A}$ is a differential subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$ of order $\theta \in (0,1]$.

Corollary 12: For every $0 < q \leq 1$ and admissible coupling weight function w, the space of matrices $\mathcal{S}_{q,w}(\mathbb{G})$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$.

We recall that the space of sparse matrices $\mathcal{S}_{0,1}$ is not inverse-closed.

Definition 13: A subalgebra $\mathcal{A}$ of a Banach algebra $\mathcal{B}$ is inverse-closed if $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{B}$ implies that $A^{-1} \in \mathcal{A}$.

Theorem 14: For $0 < q \leq 1$, suppose that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$. Then $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{G}))$.

We refer to [33] for a detailed proof of this theorem and an explicit upper bound on the q-norm of the inverse matrix. The results of Theorem 14 and Corollary 12 imply that $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q \leq 1$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{G}))$.

V. Riccati Equations over q-Banach Algebras

In this subsection, we consider solving algebraic Riccati equations related to linear-quadratic regulator (LQR) problems which are defined over q-Banach algebras. The following result is well-known and classic. Suppose that $A, Q, R \in \mathcal{B}(\ell^2(\mathbb{G}))$ and operators Q and R are positive and strictly positive on $\ell^2(\mathbb{G})$, respectively. If $(A, Q^{1/2})$ is exponentially detectable, then the algebraic Riccati equation

$$A^*X + XA - XRX + Q = 0 \tag{9}$$

has a unique strictly positive solution $X \in \mathcal{B}(\ell^2(\mathbb{G}))$. Furthermore, the matrix of the closed-loop system $A_X = A - RX$ is exponentially stable in $\mathcal{B}(\ell^2(\mathbb{G}))$. The LQR state feedback matrix is given by

$$K = -RX. \tag{10}$$

As discussed earlier, a q-Banach algebra for $0 < q < 1$ is not a Banach space. Therefore, we develop a constructive proof based on finite covering of compact sets to show that the unique solution of an algebraic Riccati equation over a q-Banach algebra belongs to that q-Banach algebra. The following result is the most general known result about structural properties of solutions of algebraic Riccati equations and all previous known results, (e.g., see [1]–[4], [10], [12], [13], [30]–[32] and references in there), turn out to be special cases of our result.

Theorem 15: For $0 < q \leq 1$, suppose that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$. Assume that $A, Q, R \in \mathcal{A}$ and matrices Q and R are positive oand strictly positive on $\ell^2(\mathbb{G})$, respectively. If $(A, Q^{1/2})$ is exponentially detectable, then the algebraic Riccati equation (9) has a unique strictly positive solution $X \in \mathcal{A}$.

For a detailed proof, we refer to [33]. According to this theorem, one can use the closure properties of $\mathcal{A}$ in order to show that the LQR state feedback gain (10) belongs to $\mathcal{A}$.

Remark 16: The proof of Theorem 15 is constructive which enables us to obtain an upper bound for the q-norm of X over each covering set. By combining these upper bounds, one can only calculate a conservative upper bound for the q-norm of X. For this reason, we will not present this conservative upper bound.

Remark 17: The results of Theorem 15 hold for $\mathcal{S}_{q,w}(\mathbb{G})$ for all $0 < q \leq 1$. For $1 \leq q \leq \infty$, $\mathcal{S}_{q,w}(\mathbb{G})$ is a Banach algebra. Our proof does not depend on whether the underlying space is a Banach space or not. Therefore, our proof works for all exponents $0 < q \leq \infty$.

VI. Sparsity Measures for Spatially Decaying Matrices

In previous sections, we show that q-norm of the unique solutions of Lyapunov and Riccati equations as well as the LQR state feedback gain (10) are bounded for all $0 < q \leq 1$. Based on our discussion in introduction section, our goal in this subsection is to propose a method to compute a reasonable value for parameter q such that $\mathcal{S}_{q,w}$–measure approximates $\mathcal{S}_{0,1}$–measure in probability.

For a given truncation threshold $\epsilon > 0$ and matrix $A = (a(i,j))_{i,j\in\mathbb{G}}$, the threshold matrix of A is denoted by $A_\epsilon = (a_\epsilon(i,j))_{i,j\in\mathbb{G}}$ and defined by setting $a_\epsilon(i,j) = 0$ if $|a(i,j)| < \epsilon$ and $a_\epsilon(i,j) = a(i,j)$ otherwise. In order to present our results in more explicit and sensible forms, only in this section, we limit our focus to the class of sub-exponentially decaying random matrices of the form

$$\mathcal{R}_{\sigma,\delta}(\mathbb{Z}) = \left\{ A = \left(r_{ij}\, e^{-\left(\frac{|i-j|}{\sigma}\right)^\delta} \right)_{i,j\in\mathbb{Z}} \middle|\, r_{ij} \sim \mathbf{U}(-1,1) \right\}$$

for some given parameters $\sigma > 0$ and $\delta \in (0,1)$. The coefficients r_{ij} are drawn from the continuous uniform distribution $\mathbf{U}(-1,1)$. It is assumed that the underlying spatial domain is $\mathbb{Z}$ and that the spatial distance between node i and j is measured by $|i-j|$. The corresponding admissible coupling weight function for this class of spatially decaying matrices is given by

$$e_{\sigma',\delta} := \left(e_{\sigma',\delta}(i,j)\right)_{i,j\in\mathbb{Z}} = \left(e^{\left(\frac{|i-j|}{\sigma'}\right)^\delta} \right)_{i,j\in\mathbb{Z}}, \tag{11}$$

for some $\sigma' > \sigma$.

Definition 18: For a given truncation threshold $0 < \epsilon < 1$, the sparsity indicator function for the class of random sub-exponentially decaying matrices $\mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ is defined by

$$\Psi_{A,w}(q,\epsilon) \;:=\; \frac{\|A\|^q_{\mathcal{S}_{q,w}}}{2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1}, \tag{12}$$

where $\lfloor . \rfloor$ is the floor function.

The sparsity indicator function provides a reasonable criterion for calculating proper values for exponent q and parameters in the weight function in order to measure sparsity of matrices in $\mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ using $\mathcal{S}_{q,w}$–measure. This is simply because of the following inequality that shows that the value of the $\mathcal{S}_{0,1}$–measure of the threshold matrix of a matrix $A \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ can be upper bounded by

$$\|A_\epsilon\|_{\mathcal{S}_{0,1}} \;\leq\; 2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1. \tag{13}$$

Theorem 19: Suppose that $w_0 \equiv 1$ is the trivial weight function and parameters $\sigma > 0$ and $\delta \in (0,1)$. Let us define parameter $\beta = q \ln \epsilon^{-1}$. For every $A \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$, the sparsity indicator function $\Psi_{A,w_0}(q,\epsilon)$ converges to

$$\gamma := \int_0^\infty e^{-\beta t^\delta}\, dt,$$

in probability as the truncation threshold ϵ tends to zero, i.e.,

$$\lim_{\epsilon \to 0^+} \; \mathbb{P}\Big\{ \; \big|\Psi_{A,w_0}(q,\epsilon) - \gamma\big| < \epsilon_0 \; \Big\} \;=\; 1, \tag{14}$$

for every $\epsilon_0 > 0$.

The result of Theorem 19 asserts that the sparsity indicator functions associated with the class of sub-exponentially decaying random matrices $\mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ can be made arbitrarily

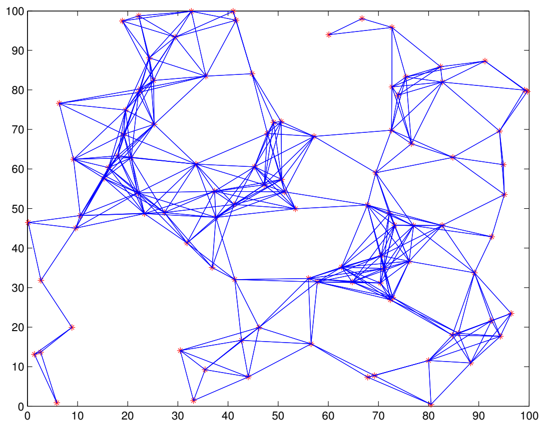

Fig. 1: The spatial configuration of a group of synchronous generators that are uniformly distributed in a box-shape region with dimension 100×100 unit square.

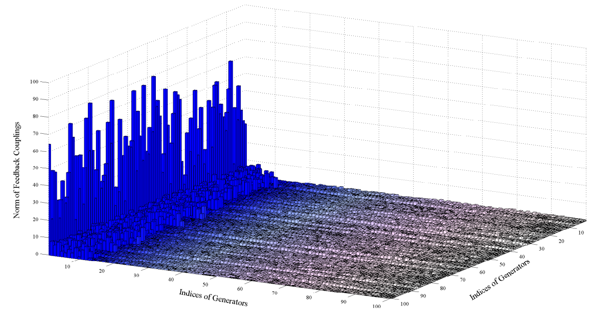

Fig. 2: This figure illustrates the spatial structure of the LQR state feedback gain for all generators. The vertical axis is showing the 2-norm of the state feedback couplings K_{ij} between every two generators. The i-axis is index of each generator. For each $i = 1, \ldots, 100$, the corresponding indices of generators on the j-axis are sorted according to their spatial distance from generator i. This is why all feedback gains with strong couplings are sorted on the left-hand-side of the plot, where the first row on the j-axis corresponds to K_{ii} for all $i = 1, \ldots, 100$.

close to a number close to one (i.e., γ) in probability when we use trivial weight function w_0 (with constant value 1). It turns out that one can select the exponent q and the nontrivial weight function w simultaneously to measure sparsity using the weighted $\mathcal{S}_{q,w}$–measure with guaranteed convergence properties.

Remark 20: Similar results can be obtained for polynomial weight functions.

VII. Simulation Results on Power Networks

We consider a power network consists of N_G synchronous generators with sparse interconnection topology (see Figure 1). The generators are randomly and uniformly distributed in a box-shape region with dimensions 100×100 unit square. The rotor dynamics of generators for purely inductive lines and constant-current loads are given by the classic second-order Kuramoto model

$$M_i \ddot{\delta}_i(t) = -D_i \dot{\delta}_i(t) + P_{G_i}(t) - \sum_{j=1}^{N_G} P_{ij} \, \sin(\delta_i(t) - \delta_j(t)),$$

for $i \in \mathbb{G} = \{1, \ldots, N_G\}$, where P_{G_i} is the effective power input to generator i and the coupling weight P_{ij} is the maximum power transferred between generators i and j which is given by $P_{ij} = E_i E_j |Y_{ij}|$. The constant E_i is the internal voltage of generator i. All angles are measured w.r.t. a 60Hz rotating frame. The reduced complex admittance matrix with entries $|Y_{ij}|$ incorporates models of transmission lines and transformers connecting generators i and j. The spatial location of generator i is denoted by $\mathbf{x}_i \in \mathbb{R}^{2\times 1}$. In order to construct a sample sparse power network, first we uniformly distribute N_G generators in the region. Then, we define the coupling structure of the network by imposing the following proximity rule: if $\|\mathbf{x}_i - \mathbf{x}_j\| > \rho$, then we set $|Y_{ij}| = 0$, otherwise $|Y_{ij}|$ can be a nonzero number drawn from the uniform distribution $\mathbf{U}(0, \mu)$ for some $\rho, \mu > 0$. The parameter $\rho > 0$ defines the proximity radius between the neighbors. The corresponding proximity graph is shown by $\mathcal{G}$ and its incidence matrix by $B(\mathcal{G})$. The vector of all angles and effective power inputs are represented by $\delta = (\delta_1, \ldots, \delta_{N_G})^*$ and $P_G = (P_{G_1}, \ldots, P_{G_{N_G}})^*$, respectively.

The centralized optimal governor control problem is to find the vector of effective power inputs for generators to enhance the steady-state security of the grid by improving the rotor angle profile, i.e., the goal is to minimize

$$J = \int_0^\infty \Big(\delta(t)^* B(\mathcal{G})^* B(\mathcal{G}) \delta(t) + P_G(t)^* P_G(t) \Big) dt.$$

In order to visualize the spatial structure of the centralized optimal state feedback controller, we linearize the swing equations around the operating point $(\bar{\delta}, \bar{\dot{\delta}}) = (0, 0)$ by replacing the nonlinear coupling terms $\sin(\delta_i - \delta_j)$ by $\delta_i - \delta_j$. The linearized swing equations is given by

$$M \ddot{\delta} \;+\; D \dot{\delta} \;+\; L \delta \;=\; 0, \tag{15}$$

where $M = \mathbf{diag}(M_1, \ldots, M_{N_G})$ and $D = \mathbf{diag}(D_1, \ldots, D_{N_G})$, and L is the Laplacian or admittance matrix with off-diagonal entries $L_{ij} = -P_{ij}$ and diagonal entries $L_{ii} = -\sum_{\substack{k=1 \\ k \neq i}}^{N_G} P_{ik}$. The centralized optimal state feedback control law for the linearized model (15) is given by

$$P_{G_i} \;=\; \sum_{j=1}^{N_G} K_{ij} \begin{pmatrix} \delta_j \\ \dot{\delta}_j \end{pmatrix}$$

where $K_{ij} \in \mathbb{R}^{2\times 1}$. The numerical simulations are done with parameters given in per unit system as follows: $D_i = M_i = 1$ and $E_i = 1$ for all $i = 1, \ldots, N_G$, $\rho = 17$, and $\mu = 5$. In our simulations, we select those sparse network samples for which $\left(\left(\begin{smallmatrix} 0 & I \\ -L & -D \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix} \right), B(\mathcal{G}) \right)$ is stabilizable and detectable. Figure 2 depicts the spatial structure of the LQR state feedback matrix $K = (K_{ij})_{i,j \in \mathbb{G}}$ and asserts that it is spatially decaying as a function spatial distance between the generators. In Theorem 15, we prove that feedback couplings

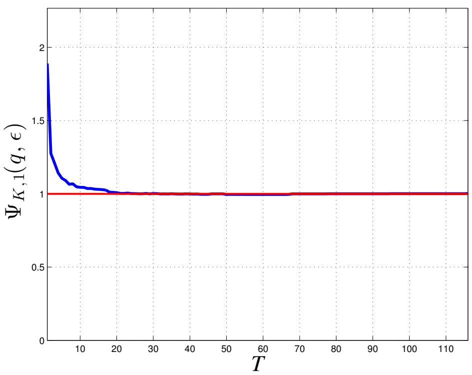

Fig. 3: This simulation shows that the sparsity of a near-optimal information structure for the randomly generated power network is about 20 and that each generator can be localized in space.

$\|K_{ij}\|_2$ cannot decay faster than sub-exponential functions for network with sparse interconnection topologies, such as a power network. In this example, our numerical calculations show that the value of the ideal sparsity measure is

$$\|K\|_{\mathcal{S}_{0,1}(\mathbb{G})} = 100, \tag{16}$$

which implies that the communication requirements in the controller array is all-to-all. The value of the ideal sparsity measure of the Laplacian matrix L is $\|L\|_{\mathcal{S}_{0,1}(\mathbb{G})} = 14$. Now, we can apply results of Theorem 19 to compute a near-optimal information structure for the power grid. In order to exploit the spatial structure of the power grid, we relate parameters ϵ (truncation threshed) and T (truncation length) by the following equation $\epsilon = \exp(-(\frac{T}{\sigma})^{\delta})$. In Figure 3, our numerical simulations show that the value of the sparsity indicator function of K w.r.t. with the trivial weight function is almost one for all $0 < q \leq 0.0137$ and $T \geq 20$. This data provides some valuable information about the sparsity of the information structure in the controller array: each generators needs to communicate with about 20 neighboring generators.

VIII. Conclusions

In this paper, we focus on an important omnipresent class of spatially distributed systems, so called spatially decaying systems, for which the corresponding optimal controllers have an inherently semi-decentralized information structures. We introduce an ideal sparsity measure that is specifically tailored to capture sparsity and spatial localization features of spatially decaying systems. A general unified mathematical framework based on q-Banach algebras are introduced to investigate notions of sparsity in spatially distributed systems.

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