Fundamental Limits and Tradeoffs on Disturbance Propagation in Linear Dynamical Networks

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***Abstract*—We investigate performance deterioration in linear consensus networks subject to external stochastic disturbances. The expected value of the steady state dispersion of the states of the network is adopted as a performance measure. We develop a graph-theoretic methodology to relate structural specifications of the coupling graph of a linear consensus network to its performance measure. We explicitly quantify several inherent fundamental limits on the best achievable levels of performance and show that these limits of performance are emerged only due to the specific interconnection topology of the coupling graphs. Furthermore, we discover some of the inherent fundamental tradeoffs between notions of sparsity and performance in linear consensus networks.**

***Index Terms*—Fundamental limits, linear consensus networks, network analysis and control, performance measures, sparsity measures.**

I. Introduction

The issue of fundamental limits and their tradeoffs in large-scale interconnected dynamical systems design lies at the very core of theory of distributed feedback control systems as it reveals what is achievable, and conversely what is not achievable by distributed feedback control laws. Improving global performance as well as robustness to exogenous disturbances in dynamical networks are critical for sustainability and energy efficiency in engineered infrastructures; examples include formation control of a group of autonomous vehicles, distributed emergency response systems, interconnected transportation networks, energy and power networks, metabolic pathways, and sociotechnical networks [1]–[8]. One of the outstanding analysis problems in the context of dynamical networks is to investigate and characterize their intrinsic fundamental limits and tradeoffs on global performance. Providing solutions to this important challenge will enable us to develop underpinning principles to design efficient-by-design dynamical networks.

In this paper, we are particularly interested in the class of first-order linear consensus networks that are driven by exogenous stochastic disturbance inputs. We quantify inherent fundamental limits on the best achievable levels of performance in such networks and show how the performance of a network in this class depends on the topology of the coupling graph. The topology of the coupling graph of a consensus network depends on the coupling structure among the subsystems, which are usually imposed by governing physical laws and/or global objectives. We consider linear consensus networks that are operating in closed-loop, i.e., networks that have been already controlled by a linear state feedback control law. In some applications such as formation control of autonomous vehicles, sparsity pattern of the underlying information structure in the controller array determines communication requirements among the vehicles, and as a result, it defines the sparsity pattern of the topology of the coupling graph of the closed-loop network.

In Section III, the steady state variance of the output of a noisy consensus network is adopted as a performance measure to quantify performance deterioration of the network. This performance measure is equal to the square of the $\mathcal{H}_2$-norm of the network from the disturbance input to the output [4]. Our first contribution shows that how the performance measure scales with the network size. For consensus networks with unweighted coupling graphs, it is shown in Section IV that the performance measure is $\Omega(n)$ for networks with "fairly" sparse interconnection topologies such as tree and unicyclic graphs,[1] where n is the network size. The performance measure scales in order of $\Omega(1)$ for networks with "fairly" dense graphs such as complete bipartite and complete graphs. In the worst case, the performance measure scales in order of $\mathcal{O}(n^2)$, where networks with path-like graphs experience the worst levels of performance. Our second contribution is to reveal the importance of the graph topology in emergence of fundamental limits on the best achievable values for the performance measure. In Section IV, we prove that by subsuming more detailed graph specifications in our calculations one can obtain tighter lower bounds for the best achievable values of the performance measure. In order to verify meaningfulness of our theoretical results, we performed extensive simulations and the results assert that our theoretical lower bounds are tighter for networks with rather dense coupling graphs (see Figs. 4–6 for more details). The impacts of the presented fundamental limits usually appear as intrinsic interplays between the performance measure and various sparsity measures in linear consensus networks. In our third contribution that is discussed in Section V, we formulate several uncertainty-principle-like inequalities that assert that networks with more sparse coupling graphs incur poorer levels of performance.

II. Mathematical Notations

Matrix Theory: The set of all nonnegative real numbers is denoted by $\mathbb{R}_+$. The $n \times 1$ vector of all ones is denoted by $\mathbf{1}_n$, the $n \times n$ identity matrix by I_n, the $m \times n$ zero matrix by $0_{m\times n}$, and the $n \times n$ matrix of all ones by J_n. We will eliminate subindices of these matrices whenever the corresponding dimensions are clear from the context. The centering matrix of size n is defined by $M_n := I_n - (1/n)J_n$. The transposition of matrix A is denoted by A^{T} and the Moore-Penrose pseudo-inverse of matrix A by $A^{\dagger}$. For a square matrix A, $\mathrm{Tr}(A)$ refers to the summation of on-diagonal elements of A. The following definitions are from [10].

[1] We employ the big omega notation in order to generalize the concept of asymptotic lower bound in the same way as $\mathcal{O}$ generalizes the concept of asymptotic upper bound. We adopt the following definition according to [9]:

$$f(n) = \Omega\left(g(n)\right) \Leftrightarrow g(n) = \mathcal{O}\left(f(n)\right) \tag{1}$$

where $\mathcal{O}$ represents the big O notation. On the left-hand side of (1), the Ω notation implies that $f(n)$ grows at least of the order of $g(n)$.

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Definition 1: For every $x \in \mathbb{R}^n_+$, let us define $x^\downarrow$ to be a vector whose elements are a permuted version of elements of x in descending order. We say that x majorizes y, which is denoted by $x \trianglerighteq y$, if and only if $\mathbf{1}^\text{T}x = \mathbf{1}^\text{T}y$ and $\sum_{i=1}^k x_i^\downarrow \geq \sum_{i=1}^k y_i^\downarrow$ for all $k = 1, \ldots, n-1$.

The vector majorization is not a partial ordering. This is because from relations $x \trianglerighteq y$ and $y \trianglerighteq x$ one can only conclude that the entries of these two vectors are equal, but possibly with different orders. Therefore, relations $x \trianglerighteq y$ and $y \trianglerighteq x$ do not imply $x = y$.

Definition 2: The real-valued function $F : \mathbb{R}^n_+ \to \mathbb{R}$ is called Schur-convex if $F(x) \geq F(y)$ for every two vectors x and y with property $x \trianglerighteq y$.

Graph Theory: Throughout this paper, we assume that all graphs are finite, simple, and undirected. A weighted graph $\mathcal{G}$ is represented by a triple $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}, w_\mathcal{G})$, where $\mathcal{V}_\mathcal{G}$ is the set of nodes, $\mathcal{E}_\mathcal{G} \subseteq \{\{i,j\} | i,j \in \mathcal{V}_\mathcal{G}, i \neq j\}$ is the set of edges, and $w_\mathcal{G} : \mathcal{E}_\mathcal{G} \to \mathbb{R}_+$ is the weight function. An unweighted graph $\mathcal{G}$ is a graph with constant weight function $w_\mathcal{G}(e) \equiv 1$ for all $e \in \mathcal{E}_\mathcal{G}$. For each $i \in \mathcal{V}_\mathcal{G}$, the degree of node i is defined by $d_i := \sum_{e=\{i,j\} \in \mathcal{E}_\mathcal{G}} w_\mathcal{G}(e)$. The sum of all edge weights in graph $\mathcal{G}$ is denoted by $W(\mathcal{G})$. The adjacency matrix $A_\mathcal{G} = [a_{ij}]$ of graph $\mathcal{G}$ is defined by setting $a_{ij} = w_\mathcal{G}(e)$ if $e = \{i,j\} \in \mathcal{E}_\mathcal{G}$, otherwise $a_{ij} = 0$. The Laplacian matrix of $\mathcal{G}$ is defined by $L_\mathcal{G} := D_\mathcal{G} - A_\mathcal{G}$, where $D_\mathcal{G} = \text{diag}(d_1, \ldots, d_n)$ is a diagonal matrix. The eigenvalues of a Laplacian matrix $L_\mathcal{G}$ are indexed in ascending order $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If $\mathcal{G}$ is connected, then $\lambda_2 > 0$. The class of all connected unweighted graphs with n nodes is denoted by $\mathbb{G}_n$ and $\mathbb{G}_n^\text{W}$ represents the set of all connected weighted graphs with n nodes. The centering graph is a complete graph with Laplacian matrix M_n and is denoted by $\mathcal{M}_n$.

For comparison purposes throughout the paper, we consider some of the standard graphs such as complete graph $\mathcal{K}_n$, star graph $\mathcal{S}_n$, cycle graph $\mathcal{C}_n$, path graph $\mathcal{P}_n$, bipartite graph $\mathcal{B}_{n_1,n_2}$, and complete bipartite graph $\mathcal{K}_{n_1,n_2}$. Every one of these graphs has its own comparable characteristics. For instance, among all graphs in $\mathbb{G}_n$ a complete graph has the maximum number of edges and a star graph has the maximum number of nodes with degree one. A path graph is a tree with minimum number of nodes of degree one. We refer to reference [11] for more details and discussions. A tree is a connected graph on n nodes with exactly $n-1$ edges. A unicyclic graph is a connected graph with exactly one cycle. A d-regular graph is a graph where all nodes have identical degree d. A subgraph $\mathcal{F}$ of a graph $\mathcal{G}$ is a spanning subgraph if it has the same node set as $\mathcal{G}$. An edge is called a cut-edge whose deletion increases the number of connected components.

For a given Laplacian matrix $L_\mathcal{G}$, the corresponding resistance matrix $R_\mathcal{G} = [r_{ij}]$ is defined using the Moore-Penrose pseudo-inverse of $L_\mathcal{G}$ by setting $r_{ij} = l^\dagger_{ii} + l^\dagger_{jj} - l^\dagger_{ji} - l^\dagger_{ij}$, where $L^\dagger_\mathcal{G} = [l^\dagger_{ij}]$. The quantity r_{ij} is so called the effective resistance between nodes i and j. Finally, the total effective resistance $\mathbf{r}_\text{total}$ is defined as the sum of the effective resistances between all distinct pairs of nodes, i.e.,

$$\mathbf{r}_\text{total} = \frac{1}{2}\mathbf{1}_n^\text{T} R_\mathcal{G} \mathbf{1}_n = \frac{1}{2}\sum_{i,j=1}^{n} r_{ij}. \tag{2}$$

III. Linear Consensus Networks and Their Performance Measures

We consider a class of first-order consensus (FOC) networks whose dynamics are defined over coupling graphs $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}, w_\mathcal{G})$ with n nodes. For this class of networks, each node corresponds to a subsystem with a scalar state variable and the interconnection topology between these subsystems is defined by the coupling graph $\mathcal{G}$. The state of the entire network is represented by $x = [x_1, x_2, \ldots, x_n]^\text{T}$ where x_i is the state variable of subsystem i for all $i = 1, \ldots, n$. The dynamics of this class of FOC networks are governed by

$$\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q}) : \begin{cases} \dot{x} = -L_\mathcal{G} x + \xi \\ y = C_\mathcal{Q} x \end{cases} \tag{3}$$

where x is the vector of state variables, ξ is an exogenous white Gaussian noise with zero-mean and identity covariance matrix, y is the performance output of the network, $L_\mathcal{G}$ is the Laplacian matrix of $\mathcal{G}$, and $C_\mathcal{Q}$ is the output matrix of the network.

Definition 3: A given graph $\mathcal{Q} = (\mathcal{V}_\mathcal{Q}, \mathcal{E}_\mathcal{Q}, w_\mathcal{Q})$ is the output graph of a linear consensus network $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$ if $\mathcal{Q}$ admits

$$L_\mathcal{Q} := C_\mathcal{Q}^\text{T} C_\mathcal{Q} \tag{4}$$

as its Laplacian matrix.

The output graph $\mathcal{Q}$ exists if the output matrix $C_\mathcal{Q}$ has zero row sums. In general, the output graph can be a disconnected graph with real-valued edge weights. The output graphs help us to better understand how the specific choice of performance output will affect a given performance measure. We adopt the following class of performance measures that are defined using the performance outputs.

Definition 4: Suppose that $\mathcal{Q}$ is an output graph of $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$. The performance measure of $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$ is defined as the steady state variance of the performance output of the network, i.e.,

$$\rho_\mathbf{ss}(L_\mathcal{G}; L_\mathcal{Q}) := \lim_{t \to \infty} \mathbb{E}\left[y(t)^\text{T} y(t)\right]. \tag{5}$$

In order to ensure that (5) is well-defined, marginally stable and unstable modes of $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$ must be unobservable from the performance output y. The following two assumptions are made for this reason.

Assumption 1: For all networks $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$ in this paper, it is assumed that $\mathcal{Q}$ is an output graph according to Definition 3.

According to this assumption, $L_\mathcal{Q}$ is the Laplacian matrix of the output graph $\mathcal{Q}$. Examples of admissible output matrices include incidence and centering matrices. When the output matrix is the centering matrix, i.e., $C_\mathcal{Q} = M_n$, the corresponding output graph is a centering graph, i.e., $\mathcal{Q} = \mathcal{M}_n$.

Assumption 2: For all $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$ networks in this paper, the corresponding coupling graph $\mathcal{G}$ is assumed to be connected.

Based on Assumption 2, one can verify that consensus network $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$ has only one marginally stable mode with eigenvector $\mathbf{1}$ and all other modes are stable. The marginally stable mode is unobservable from the performance output y, because the output matrix of the network satisfies $C_\mathcal{Q}\mathbf{1} = 0$. Therefore, the performance measure (5) is well-defined (cf. [1, Sec. III]).

The performance measure (5) quantifies the performance of the network in the average. This is because (5) is indeed equivalent to the square of the $\mathcal{H}_2$-norm of the system from the exogenous noise input to the performance output [1], [12]–[14]. When there is no exogenous noise input, the steady state of $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$ converges to the consensus state and the value of the performance measure becomes zero. In the following, we quantify performance measure (5) for the class of FOC networks.

Theorem 1: For a given network $\mathcal{N}(L_\mathcal{G}; L_\mathcal{Q})$, the performance measure (5) can be quantified as

$$\rho_\mathbf{ss}(L_\mathcal{G}; L_\mathcal{Q}) = \frac{1}{2}\mathbf{Tr}\left(L_\mathcal{Q} L_\mathcal{G}^\dagger\right) \tag{6}$$

where $L_\mathcal{G}^\dagger$ is the Moore–Penrose pseudo inverse of $L_\mathcal{G}$.

Proof: Let us define the disagreement vector by [15]

$$x_d(t) := M_n x(t) = x(t) - \frac{1}{n} J_n x(t). \tag{7}$$

By multiplying a vector by the centering matrix, we actually subtract the mean of all the entries of the vector from each entry. The dynamics of $\mathcal{N}(L_{\mathcal{G}}; L_{\mathcal{Q}})$ with respect to the new state transformation (7) is so called disagreement form of the network, which is given by

$$\mathcal{N}_d\left(L_{\mathcal{G}_d}; L_{\mathcal{Q}}\right) : \begin{cases} \dot{x}_d = -L_{\mathcal{G}_d} x_d + M_n \xi \\ y = C_{\mathcal{Q}} x_d \end{cases}$$

in which $L_{\mathcal{G}_d} = L_{\mathcal{G}} + (1/n)J_n$ and the new state matrix is indeed stable. One can easily verify that the transfer functions from ξ to y in both networks $\mathcal{N}(L_{\mathcal{G}}; L_{\mathcal{Q}})$ and $\mathcal{N}_d(L_{\mathcal{G}_d}; L_{\mathcal{Q}})$ are identical. Therefore, the $\mathcal{H}_2$-norm of the system from ξ to y in both representations are well-defined and equivalent. Let us consider the integral form of the output of network $\mathcal{N}_d(L_{\mathcal{G}_d}; L_{\mathcal{Q}})$ as follows:

$$y(t) = C_{\mathcal{Q}} \int_0^t e^{-L_{\mathcal{G}_d}(t-\tau)} M_n \xi(\tau) d\tau. \tag{8}$$

By substituting $y(t)$ from (8) in (5), calculating the expected value, and finally taking the limit, the value of the performance measure can be calculated using the trace formula $\mathrm{Tr}(P_c L_{\mathcal{Q}})$, where matrix P_c is the controllability Gramian of the disagreement network $\mathcal{N}_d(L_{\mathcal{G}_d}; L_{\mathcal{Q}})$ and it is the solution of the Lyapunov equation

$$L_{\mathcal{G}_d} P_c + P_c L_{\mathcal{G}_d} - M_n = 0.$$

Since $-L_{\mathcal{G}_d}$ is stable, the above Lyapunov equation has a unique positive definite solution [16, Th. 7.11]. Using the fact that $L_{\mathcal{G}}^{\dagger} L_{\mathcal{G}_d} = L_{\mathcal{G}_d} L_{\mathcal{G}}^{\dagger} = M_n$, we get $P_c = (1/2) L_{\mathcal{G}}^{\dagger}$ and the desired result follows. ■

If the output graph is a centering graph, then the performance measure (6) reduces to

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) = \frac{1}{2} \mathbf{Tr}\left(L_{\mathcal{G}}^{\dagger}\right) = \frac{1}{2} \sum_{i=2}^{n} \lambda_i^{-1} \tag{9}$$

where λ_i for $i = 2, \ldots, n$ are nonzero eigenvalues of $L_{\mathcal{G}}$ and $\lambda_1 = 0$ according to Assumption 2.

Remark 1: The performance measure (6) relates to the concept of coherence in consensus networks and the expected dispersion of the state of the system in steady state [1], [12]. It also has close connections to the total effective resistance of graph $\mathcal{G}$ as follows:

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) = \frac{1}{2n} \mathbf{r}_{\text{total}} \tag{10}$$

where the total effective resistance of $\mathcal{G}$ is given by $\mathbf{r}_{\text{total}} = n \sum_{i=2}^{n} \lambda_i^{-1}$; we refer to [1], [17] for more details.

IV. Fundamental Limits on the Performance Measure

We evaluate the performance of the class of FOC networks (3) with respect to the centering output graph with the following corresponding performance measure:

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) = \frac{1}{2} \sum_{i=2}^{n} \lambda_i^{-1}. \tag{11}$$

In this section, several scenarios are investigated in order to reveal the important role of the coupling graphs of FOC networks on emergence of fundamental limits on (11).

A. Universal Bounds and Scaling Laws

The following result presents universal lower and upper bounds for the best and worst achievable values for (11) among all FOC networks with arbitrary unweighted coupling graphs.

Theorem 2: For a given FOC network with an unweighted coupling graph $\mathcal{G} \in \mathbb{G}_n$, the performance measure (11) is bounded by

$$\frac{1}{2} - \frac{1}{2n} \leq \rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \frac{n^2 - 1}{12}. \tag{12}$$

Furthermore, the lower bound is achieved if and only if $\mathcal{G} = \mathcal{K}_n$, and the upper bound is reached if and only if $\mathcal{G} = \mathcal{P}_n$.

Proof: We use the result of Theorem 9 that implies that for any graph $\mathcal{G}$ with n nodes, we have $\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \geq \rho_{\mathbf{ss}}(L_{\mathcal{K}_n}; M_n)$, because graph $\mathcal{G}$ is always a subgraph of $\mathcal{K}_n$. A straightforward computation shows that $\rho_{\mathbf{ss}}(L_{\mathcal{K}_n}; M_n) = (n-1)/(2n)$. On the other hand, every connected graph $\mathcal{G}$ contains a spanning tree $\mathcal{T}$. Using Theorem 9 and the fact that $\mathcal{T}$ is a subgraph of $\mathcal{G}$, we get $\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \rho_{\mathbf{ss}}(L_{\mathcal{T}}; M_n)$. Moreover, Theorem 3 provides an upper bound for $\rho_{\mathbf{ss}}(L_{\mathcal{T}}; M_n)$, which is valid for all trees $\mathcal{T}$. Hence, this upper bound provides the desired upper bound. ■

The bounds in inequalities (12) only depend on the network size and it is assumed that nothing specific is known about the interconnection topology of the network. These bounds can be tightened if we consider more specific subclasses of graphs. In the following three theorems, we improve upon the bounds in Theorem 2 for three important classes of graphs.

Theorem 3: For a given FOC network with an unweighted tree coupling graph $\mathcal{T} \in \mathbb{G}_n$ and $n \geq 5$, the performance measure (11) is bounded by

$$\frac{(n-1)^2}{2n} \leq \boldsymbol{\rho}_{\mathbf{ss}}(L_{\mathcal{T}}; M_n) \leq \frac{n^2 - 1}{12}. \tag{13}$$

Moreover, the lower bound is achieved if and only if $\mathcal{T} = \mathcal{S}_n$ and the upper bound is achieved if and only if $\mathcal{T} = \mathcal{P}_n$.

Proof: We consider the characteristic polynomial of the Laplacian matrix of the coupling graph $\mathcal{T}$

$$\Phi_{\mathcal{T}}(\lambda) = \sum_{k=0}^{n} (-1)^{n-k} c_k(\mathcal{T}) \lambda^k. \tag{14}$$

From (9) and Vieta's formulas for (14), it follows that:

$$\rho_{\mathbf{ss}}(L_{\mathcal{T}}; M_n) = \frac{c_2(\mathcal{T})}{2c_1(\mathcal{T})}. \tag{15}$$

We also know that $c_1(\mathcal{T}) = \prod_{i=2}^{n} \lambda_i$ and it is equal to n for trees. Therefore, one can rewrite (15) as follows:

$$\rho_{\mathbf{ss}}(L_{\mathcal{T}}; M_n) = \frac{c_2(\mathcal{T})}{2n}. \tag{16}$$

One of the invariant characteristics of a graph is its Wiener number that is denoted by $W(\mathcal{T})$ [18]. This quantity is equal to the sum of distances between all pairs of nodes of $\mathcal{T}$. It is well known that the second coefficient of the Laplacian characteristic polynomial of a tree coincides with the Wiener number, i.e., $c_2(\mathcal{T}) = W(\mathcal{T})$. According to this fact and (16), it follows that:

$$\rho_{\mathbf{ss}}(\mathcal{T}) = \frac{W(\mathcal{T})}{2n}. \tag{17}$$

According to [19], if $\mathcal{T}$ is a tree with n nodes that is neither $\mathcal{P}_n$ nor $\mathcal{S}_n$, then

$$W(\mathcal{S}_n) < W(\mathcal{T}) < W(\mathcal{P}_n). \tag{18}$$

Furthermore, it is shown that [19]

$$W(\mathcal{P}_n) = \binom{n+1}{3} \quad \text{and} \quad W(\mathcal{S}_n) = (n-1)^2. \tag{19}$$

From (17), (18) and (19), we have

$$\frac{(n-1)^2}{2n} < \rho_{\mathbf{ss}}(L_{\mathcal{T}}; M_n) < \frac{n^2 - 1}{12}.$$

On the other hand, it follows from (19) and (17) that:

$$\rho_{\mathbf{ss}}(L_{\mathcal{P}_n}; M_n) = \frac{n^2-1}{12} \text{ and } \rho_{\mathbf{ss}}(L_{\mathcal{S}_n}; M_n) = \frac{(n-1)^2}{2n}.$$

Therefore, the lower bound in (13) is achieved if and only if $\mathcal{T} = \mathcal{S}_n$, and the upper bound is achieved if and only if $\mathcal{T} = \mathcal{P}_n$. ■

The lower bound in (13) implies that if the value of the performance measure for some FOC network is strictly less than $(n-1)^2/2n$, then the unweighted coupling graph of the network must contain at least one cycle. The next result quantifies tight bounds for FOC networks with exactly one cycle in their coupling graphs.

Theorem 4: For a given FOC network with an unweighted unicyclic coupling graph in $\mathbb{G}_n$ and $n \geq 13$, the performance measure (11) is bounded by

$$\frac{(n-1)^2}{2n} - \frac{1}{3} \leq \rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \frac{n^2-1}{12} + \frac{3}{2n} - 1. \tag{20}$$

Moreover, the lower bound is achieved if and only if $\mathcal{G} = \mathcal{S}(\mathcal{K}_3; \mathcal{K}_1, \ldots, \mathcal{K}_1)$, which is a star-like graph that is formed by replacing the center of $\mathcal{S}_n$ by a clique $\mathcal{K}_3$, and the upper bound is achieved if and only if $\mathcal{G} = \mathcal{P}(\mathcal{K}_3; \mathcal{K}_1, \ldots, \mathcal{K}_1)$, which is a path-like graph that is formed by replacing one of the end nodes of $\mathcal{P}_n$ by a clique $\mathcal{K}_3$.

Proof: According to (10), the performance measure (11) can be expressed based on the total effective resistance of the coupling graph $\mathcal{G}$. Moreover, the total effective resistance of a graph is the same as its Kirchhoff index. The rest of the proof is a revised version of proof of [20, Th. 4.4]. We omit the details due to space limitations. ■

The lower and upper bounds in (20) are tight, in the sense that if the value of the performance measure for a FOC network does not satisfy (20), then the coupling graph of this network is either a tree (with no cycle) or has at least two cycles. The following result investigates the performance of a FOC network with a bipartite coupling graph. In this case, the network consists of two disjoint sets of nodes and the states of one set depend on the states of the other set and vice versa. Bipartite graphs appear in several applications such as networks of electricity sellers and buyers [21, Ch. 12], power networks [22, Sec. 2], and networks of leaders and followers agents where leaders are only influenced by their followers and vice versa.

Theorem 5: For a given FOC network with an unweighted bipartite graph $\mathcal{G} \in \mathbb{G}_n$, the performance measure (11) is bounded by

$$1 - \frac{\lfloor \frac{n}{2} \rfloor}{n\lceil \frac{n}{2} \rceil} \leq \rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \frac{n^2-1}{12}.$$

Furthermore, the lower bound is achieved if and only if $\mathcal{G} = \mathcal{K}_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, and the upper bound is achieved if and only if $\mathcal{G} = \mathcal{P}_n$, where $\lfloor . \rfloor$ and $\lceil . \rceil$ are the floor and ceiling operators, respectively.

Proof: According to Theorem 2, a path graph $\mathcal{P}_n$ has the maximal level of performance measure among all graphs with n nodes. Moreover, $\mathcal{P}_n$ is in fact a bipartite graph. Therefore, we get

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \frac{n^2-1}{12}.$$

The best achievable lower bound can be obtained by some calculations from (10) and the result of [23, Th. 3.1], which provides bounds on Kirchhoff index of a bipartite graph. ■

The lower bound in Theorem 5 is tight. This is because if the value of the performance measure is strictly less than $1 - \lfloor n/2 \rfloor / (n \lceil n/2 \rceil)$ for a given FOC network with an unweighted coupling graph, then the coupling graph of the network cannot be a bipartite graph.

B. Bound Calculations via Exploiting Structure of Coupling Graphs

In the previous subsection, we derived lower and upper bounds for the performance measure of networks with unweighted graphs. These bounds are only functions of the network size. In this subsection, we incorporate additional knowledge of graph specifications in calculating lower and upper bounds for the performance measure. We consider five important graph specifications and extend our analysis for FOC networks with weighted and unweighted coupling graphs.

1) Graph Diameter and Number of Edges: The diameter of a graph is the largest distance between every pair of nodes in that graph.

Theorem 6: For a given FOC network with an arbitrary unweighted graph $\mathcal{G} \in \mathbb{G}_n$, the performance measure (11) is bounded by

$$\mathfrak{L}_{\mathcal{G}} \leq \rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \mathfrak{U}_{\mathcal{G}} \tag{21}$$

where $\mathfrak{L}_{\mathcal{G}} = (n-1)^2/(4m)$ and

$$\mathfrak{U}_{\mathcal{G}} = \frac{1}{2n}\left(n - 1 + \left[\binom{n}{2} - m\right] \operatorname{diam}(\mathcal{G})\right)$$

where $\operatorname{diam}(\mathcal{G})$ is the diameter and m is the number of edges of $\mathcal{G}$.

Proof: For the lower bound, we apply the inequality of arithmetic and harmonic means and (9)

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) = \frac{1}{2}\sum_{i=2}^{n} \lambda_i^{-1} \geq \frac{(n-1)^2}{2\sum_{i=2}^{n} \lambda_i} = \frac{(n-1)^2}{4m}.$$

On the other hand, using (10) and (2) for the upper bound, we get

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) = \frac{1}{2n}\sum_{i \neq j} r_{ij} = \frac{1}{2n}\left(\sum_{e \in \mathcal{E}_{\mathcal{G}}} r_e + \sum_{e \notin \mathcal{E}_{\mathcal{G}}} r_e\right). \tag{22}$$

Moreover, based on [24, Lemma 2] for unweighted graph we have $\sum_{e \in \mathcal{E}_{\mathcal{G}}} r_e = n - 1$. From this fact and (22), it follows that:

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) = \frac{n-1}{2n} + \frac{1}{2n}\sum_{e \notin \mathcal{E}_{\mathcal{G}}} r_e. \tag{23}$$

We note that the distance between two nodes of graph $\mathcal{G}$ is less than or equal to $\operatorname{diam}(\mathcal{G})$. Therefore, we have $r_{ij} = r_{\{i,j\}} \leq \operatorname{diam}(\mathcal{G})$. Using this fact and (23), we get the desired upper bound

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \frac{1}{2n}\left(n - 1 + \left[\binom{n}{2} - m\right] \operatorname{diam}(\mathcal{G})\right).$$

■

Remark 2: We note that a star graph $\mathcal{S}_n$ achieves the upper bound in (21), which means that among all unweighted connected graphs with $\operatorname{diam}(\mathcal{G}) = 2$ and $n-1$ links graph $\mathcal{S}_n$ has the maximal performance measure. Also If $\mathcal{G} = \mathcal{K}_n$, then the lower and upper bounds in (21) coincide and $\rho_{\mathbf{ss}}(L_{\mathcal{K}_n}; M_n) = (n-1)/(2n)$.

2) Total Weight Sum: The sum of all edge weights in a weighted graph $\mathcal{G}$ is defined by $W(\mathcal{G}) := \sum_{e \in \mathcal{E}_{\mathcal{G}}} w_{\mathcal{G}}(e)$.

Proposition 1: For a given FOC network with an arbitrary weighted coupling graph $\mathcal{G} \in \mathbb{G}_n^{\mathrm{W}}$, the performance measure (11) is bounded from below by

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \geq \frac{(n-1)^2}{4W(\mathcal{G})}. \tag{24}$$

Proof: It can be shown that $\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n)$ is a Schur-convex function with respect to $[\lambda_2, \ldots, \lambda_n]^{\mathrm{T}} \in \mathbb{R}_{++}^{n-1}$, where λ_i for $i = 2, \ldots, n$ are eigenvalues of $L_{\mathcal{G}}$. On the other hand, we have

$$\frac{\mathbf{Tr}(L_{\mathcal{G}})}{n-1}\mathbf{1}_{n-1}^{\mathrm{T}} \trianglelefteq [\lambda_2, \ldots, \lambda_n]^{\mathrm{T}}.$$

Therefore, according to the definition of Schur-convex functions, we can conclude inequality (24). ■

3) Number of Spanning Trees: A spanning subgraph of $\mathcal{G}$ is called a spanning tree if it is also a tree. The weighted number of spanning trees of a connected graph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}, w_\mathcal{G})$ is defined by

$$\mathfrak{T}(\mathcal{G}) := \sum_{\mathcal{T}} \prod_{e \in \mathcal{E}_\mathcal{T}} w_\mathcal{G}(e) \tag{25}$$

where the summation runs over all spanning trees $\mathcal{T}$ of $\mathcal{G}$. For unweighted graphs, the total number of spanning trees of a connected graph is an invariant graph specification.

Proposition 2: For a given FOC network with an arbitrary weighted coupling graph $\mathcal{G} \in \mathbb{G}_n^{\mathrm{W}}$, the performance measure (11) is bounded from below by

$$\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) \geq \frac{n-1}{2 \sqrt[n-1]{n\mathfrak{T}(\mathcal{G})}} \tag{26}$$

where $\mathfrak{T}(\mathcal{G})$ is the number of spanning trees of $\mathcal{G}$ defined by (25).

Proof: By applying the inequality of arithmetic and geometric means to (9), we get

$$\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) = \frac{1}{2}\sum_{i=2}^{n} \lambda_i^{-1} \geq \frac{n-1}{2} \sqrt[n-1]{\prod_{i=2}^{n} \lambda_i^{-1}}. \tag{27}$$

Using Kirchhoff's matrix tree theorem the number of spanning trees of graph can be expressed as $\mathfrak{T}(\mathcal{G}) = (1/n)\prod_{i=2}^{n} \lambda_i$. Then, using this fact and (27), we get the desired lower bound. ∎

The result of this proposition holds for general weighted connected graphs. However, for some particular classes of unweighted connected graphs, the total number of spanning trees can be calculated explicitly as a function of n. For example, for an unweighted complete graph $\mathcal{K}_n$ the total number of spanning trees is $\mathfrak{T}(\mathcal{G}) = n^{n-2}$. In fact, the lower bound in (26) is tight for weighted and unweighted graphs and it can be achieved by complete graphs. Nonetheless, our analysis shows that the proposed lower bound in (26) is not tight for the class of unweighted tree, cycle, and complete bipartite graphs. As we discussed earlier, our results in Section IV-A are tight for these classes of graphs.

4) Number of Cut Edges: An edge e is called a cut edge of $\mathcal{G}$ if removing e from $\mathcal{G}$ results in more than one connected component. The total number of cut edges in $\mathcal{G}$ is denoted by $\kappa(\mathcal{G})$.

Theorem 7: For a given FOC network with an arbitrary unweighted coupling graph $\mathcal{G} \in \mathbb{G}_n$ that has $\kappa(\mathcal{G})$ cut edges, the performance measure (11) is bounded from below by

$$\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) \geq \frac{1}{2n} + \frac{\kappa(\mathcal{G})+1}{2} - \frac{1}{n-\kappa(\mathcal{G})}. \tag{28}$$

The equality holds if and only if $\mathcal{G} = \mathcal{S}(\mathcal{K}_{n-\kappa(\mathcal{G})}; \mathcal{K}_1, \ldots, \mathcal{K}_1)$, i.e., $\mathcal{G}$ is a star graph that is formed by replacing the center of the star with a clique $\mathcal{K}_{n-\kappa(\mathcal{G})}$.

Proof: It is shown that the performance measure of (3) can be calculated by $\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) = \mathbf{r}_{\text{total}}/(2n)$. Moreover, in reference [25] it is shown that the $\mathbf{r}_{\text{total}}$ can be bounded from below as

$$\mathbf{r}_{\text{total}} \geq n\left(\kappa(\mathcal{G})+1\right) + 1 - \frac{2n}{n-\kappa(\mathcal{G})}$$

for all connected graphs with n nodes and $\kappa(\mathcal{G})$ cut edges. The lower bound can be achieved if and only if $\mathcal{G} = \mathcal{S}(\mathcal{K}_{n-\kappa(\mathcal{G})}; \mathcal{K}_1, \ldots, \mathcal{K}_1)$. ∎

For a given graph in $\mathbb{G}_n$, the number of cut edges satisfies $0 \leq \kappa(\mathcal{G}) \leq n-1$, where a tree with $n-1$ cut edges has the maximum and a complete graph with zero cut edge has the minimum number of cut edges among all graphs in $\mathbb{G}_n$. A simple calculation reveals that the lower bound in (28) gains its maximum value for tree and its minimum value for complete graphs. This asserts that the lower bound in (28) is tight according to the results of Theorems 2 and 3.

5) Degree Sequence: A degree sequence is a monotonic nonincreasing sequence of the node degrees of the coupling graph.

Theorem 8: For a given FOC network with an arbitrary weighted coupling graph $\mathcal{G} \in \mathbb{G}_n^{\mathrm{W}}$ and degree sequence $\{d_i\}_{i=1}^n$, the performance measure (11) is bounded from below by

$$\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) \geq \Delta(\mathcal{G}) \tag{29}$$

where

$$\Delta(\mathcal{G}) := \max_{\alpha>0}\left\{-\frac{1}{n\alpha} + \sum_{i=1}^{n}\frac{1}{2d_i+\alpha}\right\}. \tag{30}$$

For an arbitrary unweighted coupling graph $\mathcal{G} \in \mathbb{G}_n$, the quantity (30) reduces to $\Delta(\mathcal{G}) = -(1/2n) + (n-1)/(2n)\sum_{i=1}^{n}(1/d_i)$, where the equality holds if $\mathcal{G}$ is a complete graph or complete bipartite graph.

Proof: The proof is done for two different cases as follows.

Weighted Graph: Let us assume that $\tilde{L}_\mathcal{G} = L_\mathcal{G} + \alpha J_n$ and $\alpha > 0$. The eigenvalues of $\tilde{L}_\mathcal{G}$ are $n\alpha, \lambda_2, \ldots, \lambda_n$, where λ_i's are eigenvalues of $L_\mathcal{G}$. Based on Schur-Horn theorem the diagonal elements of $\tilde{L}_\mathcal{G}$ are majorized by its eigenvalues. Therefore, we have

$$\sum_{i=1}^{n}\frac{1}{d_i+\alpha} \leq \frac{1}{n\alpha} + \sum_{i=2}^{n}\lambda_i^{-1}. \tag{31}$$

From the definition of $\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n)$ and (32), it follows that:

$$\frac{-1}{n\alpha} + \sum_{i=1}^{n}\frac{1}{2d_i+\alpha} \leq \rho_{\mathbf{ss}}(L_\mathcal{G}; M_n). \tag{32}$$

Unweighted Graph: Using the same idea in the proof of Theorem 6, we can rewrite the performance measure of (3) as follows:

$$\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) = \frac{n-1}{2n} + \frac{1}{2n}\sum_{e \notin \mathcal{E}_\mathcal{G}} r_e.$$

Note that $r_{ij} = r_{\{i,j\}} \geq (1/d_i) + (1/d_j)$. This implies that

$$\begin{aligned}\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) &\geq \frac{n-1}{2n} + \frac{1}{2n}\sum_{\{i,j\} \notin \mathcal{E}_\mathcal{G}}\left(\frac{1}{d_i}+\frac{1}{d_j}\right)\\ &= \frac{-1}{2n} + \frac{n-1}{2n}\sum_{i=1}^{n}\frac{1}{d_i}.\end{aligned}$$

The interested reader is referred to [26] for similar arguments. ∎

For unweighted coupling graphs, the lower bound given by Theorem 8 is tighter than the lower bound given by Theorem 6. For d-regular weighted coupling graphs, the lower bound is $\Delta(\mathcal{G}) = (n-1)^2/(2nd)$. This lower bound is tight for FOC networks with weighted coupling graphs, in the sense that the performance measure of a FOC network with the weighted coupling graph $\mathcal{K}_n$ with identical edge weights $d/(n-1)$ meets the lower bound.

Remark 3: In Theorem 2, it is shown that the performance measure of a FOC network with an arbitrary unweighted coupling graph in $\mathbb{G}_n$ is always less than or equal to $(n^2-1)/12$. In the following, we show by means of three simple examples that the performance measure of a FOC network with a weighted coupling graph can be made arbitrarily large. We consider a FOC network with three nodes and path coupling graph. The edge weights are given by $w(\{1,2\}) = a$ and $w(\{2,3\}) = 1-a$, where $a > 0$. For different values of parameter a, the total sum of edge weights is equal to 1. However, we have $\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) \to \infty$ as $a \to 0$. Which implies that the performance measure cannot be uniformly bounded from above. Now for this graph, let us change the edge weights to $w(\{1,2\}) = a$ and $w(\{2,3\}) = a^{-1}$. According to (25), the total number of spanning trees of this graph is equal to 1. It is straightforward to verify that $\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) \to \infty$ as $a \to 0$. In the third scenario, let us consider a cyclic graph with four nodes and edge weights $w(\{1,2\}) = w(\{3,4\}) = a$ and $w(\{2,3\}) = w(\{1,4\}) = 1-a$. In this case, the weighted degree sequence is $d_1 = d_2 = d_3 = d_4 = 1$. A simple calculation shows that $\rho_{\mathbf{ss}}(L_\mathcal{G}; M_n) \to \infty$ as $a \to 0$. These examples explain why the performance measure of a FOC network with a weighted coupling graph can be arbitrarily large. For comparison purposes, the results of Theorems 2-4 and 7 are applied to different graphs and the results are explained in Figs. 1–3.

C. Interpretation of Bounds as Fundamental Limits

The value of the performance measure (5) for linear consensus network (3) is equal to the average output energy of the network when $\xi(t) = 0$ for all $t \geq 0$ and with a white Gaussian random initial condition $x(0)$ that satisfies $\mathbb{E}[x(0)x(0)^{\mathrm{T}}] = I_n$. In fact, it can be shown that

$$\rho_{\mathbf{ss}}(A; L_Q) = \mathbb{E}\left[\int_0^\infty y\left(t; x(0)\right)^{\mathrm{T}} y\left(t; x(0)\right) dt\right] \tag{33}$$

where $y(t; x(0))$ is the output of the linear dynamical network with respect to initial condition $x(0)$. This relationship enables us to equivalently interpret the performance measure (5) as the average energy needed to be consumed throughout the network in order to render the state of the randomly perturbed linear dynamical network to its equilibrium (i.e., consensus) state. Therefore, our theoretical bounds in Section IV-A and B can be viewed as quantification of inherent fundamental limits on the minimum average energy required to be dissipated in the network in order to reach the consensus state again in steady state. The use of term *fundamental* (or equivalently *hard*) limits for lower and upper bounds in Section IV-A and B is appropriate and meaningful. The reason is that according to our results, the performance measure of a linear consensus network whose coupling graph has some known graph specification (e.g., number of nodes, number of spanning trees, total sum of edge weights, degree sequence, etc.) cannot be better and worse than our theoretical lower bounds and upper bounds, respectively. The philosophy behind our several results presented in Section IV-B can be explained by portraying the value of performance measure for FOC networks versus various known graph specifications. In order to conceptualize the idea, we only focus on three graph specifications in our analysis. Figs. 4–6 depict the value of the performance measures for FOC networks with coupling graphs in $\mathbb{G}_7^{\mathrm{W}}$. In these figures, the points with star markers correspond to performance measures of all FOC networks with unweighted graphs in $\mathbb{G}_7$. The total number of such networks are 1,866,256. In all three figures, the gray shaded area above the red dashed curve corresponds to performance measures of FOC networks with weighted coupling graphs. In Fig. 4, the performance measure (11) is drawn for different values of weight sum $W(\mathcal{G})$. The lower bound in (24) is highlighted by a red dashed curve and it draws a fundamental limit on the best achievable performance measures. One observes that the lower bound in (24) is tight for a given value of weight sum. In fact, for a given $W(\mathcal{G})$ there exists a weighted graph with total weight sum $W(\mathcal{G})$ whose performance measure reaches the exact value of the fundamental limit $(n-1)^2/(4W(\mathcal{G}))$, where in this simulation $n = 7$. However, this lower bound is loose for unweighted graphs. For unweighted graphs, the weight sum is equal to the total number of edges in the coupling graph and it only assumes integer values. By exhausting all possible choices for unweighted graphs with identical number of edges in Fig. 4, we show that there is a gap between the actual best achievable lower bound and our theoretical fundamental limit in (24). It can be perceived that this gap is smaller for denser coupling graphs. This observation suggests that our theoretical fundamental limit in (24) is looser for sparse coupling graphs and have tighter gaps for dense coupling graphs. Nevertheless, having more detailed knowledge about graph specifications helps to close the gap. For example, the weight sums for FOC networks with tree and unicyclic coupling graphs are equal to 6 and 7, respectively. In these cases, the actual minimum and maximum achievable values of performance measure exactly matches with our theoretical fundamental limits in (20) and (13).

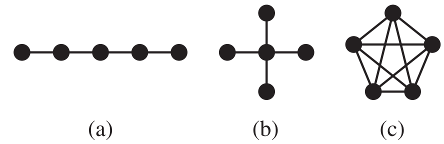

Fig. 1. This figure illustrates results of Theorems 2 and 3 for the following extreme cases. The performance measure (11) is (a) maximal for $\mathcal{P}_5$ among all graphs as well as among all trees in $\mathbb{G}_5$, (b) minimal for $\mathcal{S}_5$ among all trees in $\mathbb{G}_5$, and (c) minimal for $\mathcal{K}_5$ among all graphs in $\mathbb{G}_5$.

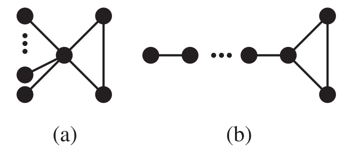

Fig. 2. Unicyclic graphs that achieve the lower and upper bounds in Theorem 4: (a) $\mathcal{G} = \mathcal{S}(\mathcal{K}_3; \mathcal{K}_1, \ldots, \mathcal{K}_1)$, and (b) $\mathcal{P}(\mathcal{K}_3; \mathcal{K}_1, \ldots, \mathcal{K}_1)$.

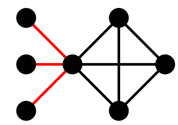

Fig. 3. Schematic graph of $\mathcal{S}(\mathcal{K}_4; \mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_1)$ that has the minimal value of performance measure among all graphs in $\mathbb{G}_7$ with exactly three cut edges (highlighted by red color).

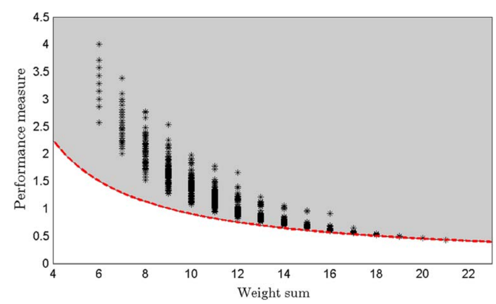

Fig. 4. The gray shaded area depicts the value of the performance measure for all FOC networks with coupling graphs in $\mathbb{G}_7^{\mathrm{W}}$ and star markers correspond to performance measures of all FOC networks with unweighted graphs in $\mathbb{G}_7$. The red dashed curve portrays the lower bound in (24).

To summarize our discussion in this part, one can also set out similar arguments for Figs. 5 and 6 to infer that our theoretical fundamental limits in Section IV-B are looser for "fairly" sparse coupling graphs and have tighter gaps for dense coupling graphs. As we discussed in Section IV-A, one can exploit the structural properties of networks with sparse coupling graphs (e.g., trees and unicyclics) to quantify tight fundamental limits.

V. Fundamental Tradeoffs Between Notions of Sparsity and the Performance Measure

One of the design objectives for large-scale linear consensus networks is to optimize network coherence by designing a coupling graph that has the best possible sparsity and locality features. A fundamental property of performance measures (11) is that they are monotonically decreasing functions of the coupling graphs in the cone of positive semidefinite matrices. This property implies that the value of the performance measure increases by sparsifying the coupling graph, which is consistent with our results in Section IV-A. In this section, we quantify fundamental tradeoffs between the performance measure (11) and sparsity measures of FOC networks. The results of the following theorem assert that the performance of a

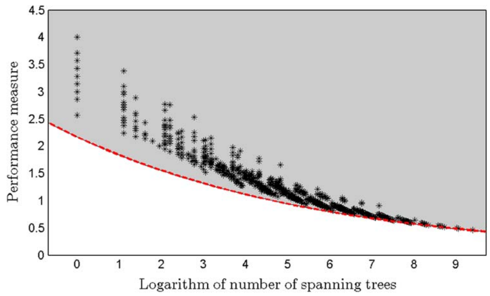

Fig. 5. The gray shaded area depicts the value of the performance measure for all FOC networks with coupling graphs in $\mathbb{G}_7^{\mathrm{W}}$ and star markers correspond to performance measures of all FOC networks with unweighted graphs in $\mathbb{G}_7$. The red dashed curve depicts the lower bound in (26).

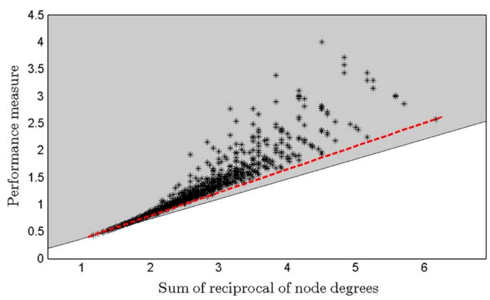

Fig. 6. The gray shaded area depicts the value of the performance measure for all FOC networks with coupling graphs in $\mathbb{G}_7^{\mathrm{W}}$ and star markers correspond to performance measures of all FOC networks with unweighted graphs in $\mathbb{G}_7$. The red dashed curve outlines the lower bound in (29) for unweighted graphs.

spanning subnetwork of a given FOC network never outperforms the performance of the parent network.

Theorem 9: Suppose that $\mathcal{G} \in \mathbb{G}_n^{\mathrm{W}}$ is the coupling graph of a given FOC network. If $\mathcal{F}$ is a connected spanning subgraph of $\mathcal{G}$, then

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \leq \rho_{\mathbf{ss}}(L_{\mathcal{F}}; M_n) \tag{34}$$

and the equality holds if and only if $\mathcal{G} = \mathcal{F}$.

Proof: Since graph $\mathcal{F}$ is a subgraph of graph $\mathcal{G}$, we have the following inequality for every $x \in \mathbb{R}^n$:

$$\begin{aligned} x^{\mathrm{T}} L_{\mathcal{G}} x &= \sum_{e=\{i,j\}\in\mathcal{E}_{\mathcal{G}}} w(e)(x_i - x_j)^2 \\ &\geq \sum_{e=\{i,j\}\in\mathcal{E}_{\mathcal{F}}} w(e)(x_i - x_j)^2 = x^{\mathrm{T}} L_{\mathcal{F}} x. \end{aligned} \tag{35}$$

This inequality implies that $L_{\mathcal{F}} \leq L_{\mathcal{G}}$, or equivalently we have $L_{\mathcal{G}}^{\dagger} \leq L_{\mathcal{F}}^{\dagger}$. From the linearity property of the trace operator and the fact that $L_{\mathcal{F}}^{\dagger} - L_{\mathcal{G}}^{\dagger}$ is a positive semi-definite matrix, we get

$$\begin{aligned} \frac{1}{2}\mathbf{Tr}\left(L_{\mathcal{F}}^{\dagger} - L_{\mathcal{G}}^{\dagger}\right) &= \frac{1}{2}\mathbf{Tr}\left(L_{\mathcal{F}}^{\dagger}\right) - \frac{1}{2}\mathbf{Tr}\left(L_{\mathcal{G}}^{\dagger}\right) \\ &= \rho_{\mathbf{ss}}(L_{\mathcal{F}}; M_n) - \rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) \geq 0. \end{aligned}$$

This completes the proof. ■

The result of this theorem implicitly asserts that adding new edges to the coupling graph of a consensus network may improve the global performance of the network. In the following, we identify several Heisenberg-like inequalities that quantify inherent fundamental tradeoffs between global performance and sparsity in FOC networks. First, we consider the following sparsity measure:

$$\|A_{\mathcal{G}}\|_0 := \mathrm{card}\left\{a_{ij} \neq 0 | A_{\mathcal{G}} = [a_{ij}]\right\} \tag{36}$$

where $A_{\mathcal{G}}$ is the adjacency matrix of the coupling graph $\mathcal{G}$. For a given graph, the value of this sparsity measure is equal to twice the number of the edges.

Theorem 10: For a given FOC network with an arbitrary unweighted coupling graph $\mathcal{G} \in \mathbb{G}_n$, there is a fundamental tradeoff between the performance measure (11) and the sparsity measure (36) that is characterized in the multiplicative form by the following inequality:

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n)\|A_{\mathcal{G}}\|_0 \geq \frac{(n-1)^2}{2} \tag{37}$$

and in the additive form by

$$\frac{\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) - \frac{1}{2} + \frac{1}{2n}}{\mathrm{diam}(\mathcal{G})} + \frac{\|A_{\mathcal{G}}\|_0}{4(n-1)} \leq \frac{n}{4}. \tag{38}$$

Let us consider the class of networks with identical number of nodes and compare several scenarios. The inequality (37) asserts that the best achievable values of performance measure (11) for sparse FOC networks are comparably larger (worse) with respect to less sparse FOC networks. For all FOC networks with identical diameters, inequality (38) implies that networks with more edges have smaller (better) values of performance measures. Among all FOC networks with identical number of edges, the ones with larger diameters can assume larger (worse) values of performance measures.

Theorem 11: Let us consider the class of FOC networks with arbitrary unweighted coupling graphs in $\mathbb{G}_n$ and a given desired performance level ρ_{ss}^*. Then, the sparsity measure (36) for this class of networks satisfies

$$\frac{(n-1)^2}{2\rho_{\mathrm{ss}}^*} \leq \|A_{\mathcal{G}}\|_0 \leq (n-1)\left[n - 4\left(\frac{\rho_{\mathrm{ss}}^* - \frac{1}{2} + \frac{1}{2n}}{\mathrm{diam}(\mathcal{G})}\right)\right]. \tag{39}$$

The result of this Theorem states that the graph diameter can be employed as a design parameter to achieve a desirable level of performance and sparsity.

The second sparsity measure that we consider in this section is so called $\mathcal{S}_{0,1}$-measure and defined by

$$\|A_{\mathcal{G}}\|_{\mathcal{S}_{0,1}} := \max\left\{\max_{1\leq i\leq n}\|A_{\mathcal{G}}(i,.)\|_0, \max_{1\leq j\leq n}\|A_{\mathcal{G}}(.,j)\|_0\right\}$$

where $A_{\mathcal{G}}(i,.)$ represents the i'th row and $A_{\mathcal{G}}(.,j)$ the j'th column of adjacency matrix $A_{\mathcal{G}}$. The value of the $\mathcal{S}_{0,1}$-measure of a matrix is the maximum number of nonzero elements among all rows and columns of that matrix [27]. The $\mathcal{S}_{0,1}$-measure of adjacency matrix of an unweighted graph is equal to the maximum node degree. The following result quantifies an inherent tradeoff between the performance measure and this sparsity measure.

Theorem 12: For a given FOC network with an arbitrary unweighted coupling graph $\mathcal{G} \in \mathbb{G}_n$ and $n \geq 3$, there is a fundamental tradeoff between the performance measure (11) and the $\mathcal{S}_{0,1}$-measure that is characterized by

$$\left(\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n) + \frac{1}{2n}\right)\|A_{\mathcal{G}}\|_{\mathcal{S}_{0,1}} \geq \frac{n-1}{2}. \tag{40}$$

The value of the $\mathcal{S}_{0,1}$-measure reveals some valuable information about sparsity as well as the spatial locality features of a given adjacency matrix, while sparsity measure (36) only provides information about sparsity. The inequality (40) asserts that the best achievable levels of performance measure (11) decreases by improving local connectivity in the coupling graph of a FOC network.

The third sparsity measure of our interest for the class of FOC networks with unweighted coupling graphs is defined by

$$\sigma(\mathcal{G}) := \max_{i,j\in\mathcal{V}_{\mathcal{G}}}\left\{\mathrm{card}\left\{\mathfrak{N}(i) \cup \mathfrak{N}(j)\right\}\right\} \tag{41}$$

where $\mathfrak{N}(i)$ is the set of all nodes that are connected to node i by an edge. The value of the sparsity measure $\sigma(\mathcal{G})$ is equal to the maximum number of nodes that are connected to any pair of nodes among all pairs of nodes in the graph. It is easy to verify that $\sigma(\mathcal{G}) \leq n$. The following result quantifies an inherent tradeoff between the performance measure and this sparsity measure.

Theorem 13: For a given FOC network with an arbitrary unweighted coupling graph $\mathcal{G} \in \mathbb{G}_n$ and $n \geq 3$, there is a fundamental tradeoff between the performance measure (11) and sparsity measure (41) that is quantified by

$$\rho_{\mathbf{ss}}(L_{\mathcal{G}}; M_n)\sigma(\mathcal{G}) \geq \frac{n-1}{2}. \tag{42}$$

Moreover, the equality holds if $\mathcal{G} = \mathcal{K}_n$.

Proof: Based on the inclusion-exclusion principle, we have

$$\text{card}\{\mathfrak{N}(i) \cup \mathfrak{N}(j)\} = d_i + d_j - \text{card}\{\mathfrak{N}(i) \cap \mathfrak{N}(j)\} \tag{43}$$

where d_i and d_j are degrees of node i and node j, respectively. Using (41) and (43), it follows that:

$$\sigma(\mathcal{G}) = \max_{\substack{i,j \in \mathcal{V}_{\mathcal{G}} \\ i \neq j}} \{d_i + d_j - |\mathfrak{N}(i) \cap \mathfrak{N}(j)|\}.$$

Then, according to [28] for the maximum eigenvalue of $L_{\mathcal{G}}$ we have $\lambda_n \leq \sigma(\mathcal{G})$. By combining this inequality and (9), we get the desired lower bound. ■

To summarize our results in this section, we conclude that there are intrinsic fundamental tradeoffs between the two favorable design objectives in linear consensus networks: minimizing the performance measure and sparsifying the coupling graph.

VI. Discussion

Several relevant network synthesis problems can be formulated in order to optimize the performance measure of a linear consensus network. There has been some recent work in this area, such as [5], [29], [30]. Some of these design problems are inherently combinatorial and intractable. For instance, problems of minimizing the performance measure by rewiring a given network with fixed number of edges or by adding a few new edges to the network are generally NP-hard problems. Therefore, having some meaningful estimates for the best achievable values of the performance measure is helpful in evaluating the efficiency of a proposed approximate algorithm to solve such non-convex and generally intractable design problems. Our lower and upper bounds in this paper provide sensible estimates for the best achievable values of the performance measure as a function of graph specifications. Moreover, if we consider the network size as design parameter, our results in Section IV-A show how rapidly the performance of a linear consensus network deteriorates as the size of network grows larger.

One observes that the performance measure (11) has several interesting functional properties. This measure is a convex function of Laplacian eigenvalues and monotonically decreasing in the space of Laplacian matrices of all connected graphs. The results of Section V highlight the importance of monotonicity property by quantifying inherent fundamental tradeoffs between sparsity and performance. A promising research direction is to investigate whether these functional properties can be used to categorize larger classes of admissible performance measures for linear consensus networks [29].

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